# Conjugate Paretian Inefficiency Measurement

Robert G. Chambers [1]

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[1]University of Maryland

# 1 Introduction

Since their inceptions, operations research and economics have studied efficiency. Debreu (1951) early elaborated an analytical framework, and a series of subsequent studies developed tractable measurement algorithms (Farrell 1957, Nerlove 1965, Afriat 1972, Hanoch and Rothschild 1972, Charnes, Cooper, and Rhodes 1978, Färe and Lovell 1978, Banker, Charnes, and Cooper 1984). Charnes et al. (1978) marked an important watershed. It showed that the Farrell-Afriat measurement schemes generalized to multiple-output settings solvable via linear programming, and its generalization by Banker et al. (1984) established *data-envelopment analysis* (DEA) as a canonical measurement framework.

Measuring efficiency requires a benchmark *efficient set* or *efficient frontier*. Here, efficiency measurement melds with multiple-criteria (vector) optimization studies that seek to optimize *vector-valued* criterion functions. With intellectual roots tracing to Edgeworth (1881) and Pareto (1909), multiple-criteria problems often invoke the Paretian criteria to identify an efficient set with the *maximal set* for a subset of the partially ordered set $\left(\mathbb{R}^S, \leq\right)$, where $\leq$ is the canonical less than or equal to partial ordering (Ehrgott 2005, Löhne 2011). DEA studies often define their benchmark efficient set using the same definition, as articulated formally in Banker et al.'s (1984) *Inefficiency Postulate* (Banker et al. 1984, Charnes, Cooper, Golany, Seiford, and Stutz 1985, Ray 2004, Pastor, Lovell, and Aparicio 2012, Russell and Schworm 2009 and 2017).

Debreu (1951, p. 273) cast efficiency measurement as measuring how far a nonoptimal situation "...is from being optimal". Later Afriat (1972, p. 576) suggested representing "...operations as nearly efficient as possible". A large literature adopts this perspective and seeks measures that judge outcomes as favorably as possible. Here, efficiency measurement melds with the study of minimum-distance problems. The Minimum Norm Duality Theorem (Nirenberg 1961, Luenberger 1969, Theorem 5.13.1) states that the minimal distance, for a given norm, between a point and a convex set's boundary is the maximal distance between the point and the set's support function with *dual variates restricted to the dual-norm's unit disc*. Rather than restricting dual variates to the unit disc, inefficiency measurement requires them to fall, variously, on a hyperplane, in a closed half space, or in the intersection of closed half spaces (for example, Debreu 1951, Charnes et al. 1978, Luenberger 1992, Ray 2007, Pastor et al. 2012).

We study efficiency measurement for a closed convex feasible set that generalizes the canonical DEA frameworks. We identify its Paretian (efficient) frontier with the maximal set for a subset of the partially ordered set $(\mathbb{R}^S, \preceq_C)$, where $\preceq_C$ denotes a *generalized inequality* that partially orders $\mathbb{R}^S$ and generalizes the canonical $\leq$ partial ordering. We seek measures that judge outcomes as favorably as possible and combine the normalization strategies pursued in the efficiency-measurement and minimum-norm literatures under a common rubric by restricting dual variates to a nonempty closed convex set. These generalizations yield an inefficiency measure defined as the conjugate (Legendre-Fenchel transform) of a closed convex function of dual variates.

In what follows, we first define notation and recall some concepts from convex analysis. Then we set up the model and use vector-optimization theory results (Löhne 2011) to define an efficient frontier using $\preceq_C$ and to characterize it using dual methods (Proposition 1). Following Debreu (1951) and Nerlove (1965), we identify a *Nerlovian* inefficiency measure and use it to define a *Paretian* inefficiency measure as the solution to a convex programming problem. We show that the Paretian inefficiency measure forms a dual conjugate pair with a restricted Nerlovian efficiency measure (Proposition 2). Then we use the conjugacy between the infimal-convolution and addition operations to show that the dual conjugacy generalizes the Nirnberg minimum-norm-minimal-distance duality. We use those results to develop: conditions that ensure that the Paretian inefficiency measure is an exhaustive function (cardinal) representation of the feasible set (Proposition 3); and composition rules for different restrictions on the feasible set and dual-variate normalization (Proposition 4). Special cases include generalizations of many inefficiency measures familiar from a DEA setting.

We examine the decomposition of measured Nerlovian inefficiency into a *technical-inefficiency* component and a *dual-inefficiency* (allocative, price) component. We show that the dual-inefficiency measure is a closed convex bi-function in the sense of Rockafellar (1970, Section 29). We discuss the dual inefficiency measure, show its relevance for recent concerns raised about Nerlovian inefficiency decompositions, and show that it satisfies two dual conjugacies: a) one to a difference-based transformation of our Paretian inefficiency measure (Proposition

5); and b) one to a difference-based transformation of Nerlovian inefficiency (Proposition 6).

Although we frame the analysis in more general terms than the polyhedral DEA setting, our results echo its familiar message that different choice criteria and different dual-normalization rules yield different efficiency measures. For example, Banker et al. (1984) derive input-oriented and output-oriented inefficiency measures by choosing different normalization criteria. Chambers, Chung, and Färe (1998) showed that the dual-variate normalization strategy of Luenberger (1992) yields inefficiency measures expressed in difference rather than ratio form. Thus, taxonomies often classify measures according to the functional structure (additive or multiplicative loss measure) of the criterion function or the orientation in which the outcome is compared to the benchmark set (slacks-based or path-based). Our results show that many of these different forms can be gathered under a common rubric that clarifies the essential mathematical issues yielding perceived differences. And because our results are established in a more general setting than DEA, it broadens the range of available measures while also showing that DEA-specific results established have implications for broader classes of measurement problems.

# 2 Notation and Preliminaries

Let $\bar{\mathbb{R}} = [-\infty, \infty]$. The *effective domain* for $f : \mathbb{R}^S \to \bar{\mathbb{R}}$, *dom* $f$, is

$$dom\ f \equiv \left\{ x \in \mathbb{R}^S : f(x) < \infty \right\},$$

and its *subdifferential correspondence*, $\partial f : \mathbb{R}^S \rightrightarrows \mathbb{R}^{S*}$, is[1]

$$\partial f(x) \equiv \left\{ q \in \mathbb{R}^{S*} : q'(z - x) \leq f(z) - f(x), \ \forall z \in \mathbb{R}^S \right\}.$$

The (convex) *conjugate* for $f$, $f^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, is[2]

$$f^*(q) \equiv \sup_{x \in \mathbb{R}^S} \left\{ q'x - f(x) \right\} \quad (1)$$

[1] $\mathbb{R}^S$ is, of course, self-dual. We retain the notation, $\mathbb{R}^{S*}$, for its dual space to ensure a clear distinction between dual and primal variates.

[2] Moreau (1966) calls $f^*$ *la fonction polaire* to $f$. Some writers call it the *Fenchel transform*. Rockafellar and Wets (2009) call it the *Legendre-Fenchel transform*.

$f^*$ is closed[3] and convex. For $f$ proper[4] closed and convex :

$$f^{**}(x) \equiv \sup_{q\in\mathbb{R}^{S*}} \{q'x - f^*(q)\} \tag{2}$$
$$= f(x),$$

and $f^*$ is also proper. Expressions (1) and (2) form the *conjugacy correspondence*, $f \overset{*}{\longleftrightarrow} f^*$, between proper closed convex $f$ and its conjugate $f^*$. A well-known consequence is (see, for example, Rockafellar 1970, Moreau 1966, Rockafellar and Wets 2009, Aubin and Ekeland 2007, Bertsekas 2009)

**Lemma 1.** *Let $f$ be proper closed convex. Then $f^*$ is proper closed convex,*

$$f(x) + f^*(q) \geq q'x \;\; \forall q, x \quad \text{(Fenchel's Inequality)} \tag{3}$$

$$\hat{q} \in \partial f(\hat{x}) \Leftrightarrow \hat{x} \in \partial f^*(\hat{q}) \Leftrightarrow f(\hat{x}) + f^*(\hat{q}) = \hat{q}'\hat{x}, \tag{4}$$

$$\partial f^*(q) = argmax_x \{q'x - f(x)\} \quad \partial f(x) = argmax_q \{q'x - f^*(q)\}. \tag{5}$$

Let $ri\ X$ denote the relative interior of $X \subset \mathbb{R}^S$. Define the *indicator function*, $\delta : \mathbb{R}^S \to \{0, \infty\}$, for $X \subset \mathbb{R}^S$ by

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ \infty & \text{otherwise.} \end{cases} \tag{6}$$

For $X \subset \mathbb{R}^S$ closed convex and nonempty, $\delta(x|X)$ is proper closed and convex. The *support function*, $\delta^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, for $X$ is

$$\delta^*(q|X) \equiv \sup\{q'x : x \in X\} \tag{7}$$
$$= \sup_{x\in\mathbb{R}^S} \{q'x - \delta(x|X)\}.$$

$\delta^*$, as the conjugate of $\delta$, is closed and sublinear. If $X$ is closed nonempty and convex, expression (2) implies $\delta^*$ is proper and

$$\delta(x|X) = \sup_{q\in\mathbb{R}^{S*}} \{q'x - \delta^*(q|X)\} \tag{8}$$

[3] A function is closed if its closure is the function itself. For proper convex functions, closedness is equivalent to lower semi-continuity (Rockafellar 1970, p. 52).

[4] A convex function $f$ is *proper* if *dom* $f$ is nonempty and $f(x) > -\infty$ for all $x$.

The *gauge function* for $X \subset \mathbb{R}^S$, $\gamma : \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined

$$\gamma(x|X) = \inf\{\gamma > 0 : x \in \gamma X\}. \tag{9}$$

If $X$ is nonempty closed convex and $0 \in X$,

$$X = \{x : \gamma(x|X) \leq 1\}.$$

The *negative polar cone*[5] of a convex set $X \subset \mathbb{R}^S$ is

$$X^* \equiv \left\{q \in \mathbb{R}^{S*} : q'x \leq 0, \forall x \in X\right\}.$$

For $X$ a closed convex nonempty cone $X^{**} = X$. The *polar set* of $X \subset \mathbb{R}^S$ is

$$X^o \equiv \left\{q \in \mathbb{R}^{S*} : q'x \leq 1, \forall x \in X\right\}.$$

The *recession (asymptotic) cone* of $X \subset \mathbb{R}^S$ is

$$X_\infty \equiv \left\{d \in \mathbb{R}^S : X + \beta d \subset X, \beta \geq 0\right\}.$$

$0 \in X_\infty$ for all $X \subset \mathbb{R}^S$. For $X$ closed convex and nonempty :

$$\begin{aligned} cl\ dom\ \delta^*(\cdot|X) &= X_\infty^* \\ (cl\ dom\ \delta^*(\cdot|X))^* &= X_\infty \end{aligned} \tag{10}$$

(Hiriart-Urruty and LeMaréchal 2001, Proposition C.2.2.4). And for $X$ closed convex with $0 \in X$:

$$X_\infty = \left\{x \in \mathbb{R}^S : \gamma(x|X) = 0\right\}$$

(Hiriart-Urruty and LeMaréchal 2001, Theorem C.1.2.5).

A useful separation result is:

**Lemma 2.** *(Rockafellar 1970) Let $X^1$ and $X^2$ be nonempty convex subsets of $\mathbb{R}^S$ that satisfy $ri\ X^1 \cap ri\ X^2 = \emptyset$, then there exists $q \in \mathbb{R}^{S*}$ such that*

$$\begin{aligned} \inf\left\{q'x : x \in X^1\right\} &\geq \delta^*\left(q|X^2\right) \\ \delta^*\left(q|X^1\right) &> \inf\left\{q'x : x \in X^2\right\}. \end{aligned}$$

[5] Some authors refer to $X^*$ as the polar cone of $X$.

# Efficient Outcomes

## The Feasible Set

Feasible outcomes are given by a closed nonempty convex $Z \subset \mathbb{R}^S$. $Z$ admits different interpretations including, among others: an input set, an output set, a technology set, a state-contingent technology set, and a (convex) envelope of observed data points. For purposes of a concrete discussion we refer to $Z$ as the *technology.* Elements of $Z$ are *netputs.*

Working with netputs differs from studies that segregate inputs from outputs. The notational difference promotes simplicity and generality and accommodates the presence of intermediate outputs in the technology. It also avoids the practical difficulties encountered in applications in segregating inputs from outputs when one decisionmaker is a net producer of, say, "corn and hogs" and another facing the same $Z$ is a net user of "corn" and produces only " hogs".

## The Efficient Set

Inefficiency-measurement and vector-optimization studies often use the canonical $\leq$ partial ordering of $\mathbb{R}^S$ (for example, Banker et al. 1984, Charnes et al. 1985, Ehrgott 2005, Ray 2004, Pastor et al. 2012, Russell and Schworm 2009 and 2017) and the Paretian criterion to identify the efficient frontier (see, for example, Banker et al.'s (1984, p. 1081) *Inefficiency Postulate*). That choice limits applicability of the resulting measures and conflicts with physical reality in many applied settings. For example, it rules out bounded $Z$ including the Charnes et al. (1985) *empirical production set* (their expression 3.1), well-documented instances of input or output congestion, the presence of by-products, and can create material-balance specifications that contradict the first law of thermodynamics.

We use instead the more general binary relation, $\preceq_C$, defined by

$$x \preceq_C y \Leftrightarrow x - y \in C,$$

where $C \subset \mathbb{R}^S$ is a closed pointed convex cone. $\preceq_C$ is reflexive, transitive, and antisymmetric so that $\left(\mathbb{R}^S, \preceq_C\right)$ forms a *partially ordered set* (Boyd and Vandenberghe 2004, Nemirovski 2007, Löhne 2011).

To relate this partial ordering to $Z$, we posit an axiom:

**Axiom 1.** $Z_\infty = C$ *(where $C \subset \mathbb{R}^S$ defines $\preceq_C$).*

When $Z$ is interpreted as a technology set, its recession cone, $Z_\infty$, describes the directions in which starting at a feasible netput in $Z$, one can move toward infinity while maintaining feasibility. Thus, it accords with the production-theoretic notion of netput-disposability.

**Example 1.** *Let $C = \mathbb{R}^S_-$. Then $Z$ satisfies the canonical Banker et al. (1984) Inefficiency Postulate (free disposability of netputs).*

**Example 2.** *Let $C = \{0\}$. Then $Z$ is compact. Special cases include the (compact) weak-disposable, convex hull technologies and the empirical production set of Charnes et al. (1985).*

**Example 3.** *Let $C = \{d\}$ where $d \in \mathbb{R}^S$. Then $Z$ satisfies the "goodness in the numeraire (d)" in the direction d criterion (Chambers and Färe 2022).*

Using Axiom 1 and the Paretian criterion, we define the efficient set as:[6]

**Definition 1.** *The efficient subset, $EffZ$, of $(Z, \preceq_C)$ is*

$$EffZ \equiv \{z^o \in Z : \nexists z \in Z \text{ for which } z^o \preceq_C z \wedge z \neq z^o\}.$$

Because $Z$ is closed convex, $\delta^*(q|Z)$ is closed sublinear. Thus, Axiom 1 and (10) imply that $dom\ \delta^*(\cdot|Z) = C^*$. We use that observation to state a result that extends those for dual representations of efficient sets for the canonical $\leq$ partial ordering (for example, Charnes et al. 1985, Ehrgott 2005, Theorem 3.6 and Corollary 3.7) to $\preceq_C$:[7]

**Proposition 1.** *a) $\partial\delta^*(q|Z) \subset EffZ$ for all $q \in ri\ C^*$. b) $z^o \in EffZ \Rightarrow z^o \in \partial\delta^*(q|Z)$ for some $q \in ri\ C^*$.*

Proof: See Appendix.

[6] Alternatively, $z^o$ is efficient if and only if no $z \in Z$ exists for which $z^o - z \in C \setminus \{0\}$.

[7] Proposition 1 can be inferred, for example, from Theorems 4.1 and 4.2 in Löhne (2011). We present a direct proof, which follows standard arguments, in an Appendix to ensure a self-contained treatment. Note the obvious connection to the First and Second Welfare Theorems of Economics.

**Remark 1.** *Proposition 1.a remains true for general closed $Z$, but part b) requires convexity. A geometric interpretation of Proposition 1 is that $EffZ$ corresponds to the set of faces of $Z$ exposed by $q \in ri\ C^*$.*

In an economic setting where competitive firms maximize profit, the connection between $\partial\delta^*(q|Z)$, as the profit-maximizing netput vectors, and $Eff\ \ Z$ is familiar. In a broader context, the connection between $\partial\delta^*(q|Z)$ and $Eff\ \ Z$ helps explain the primacy of linear scalarization techniques in solving vector-optimization problems (Ehrgott 2005, Löhne 2011).

# Inefficiency Measures

## Measures Defined

We call

$$q'z - \delta^*(q|Z) = \delta^*(q|z - Z)$$

the *q-Nerlove efficiency measure for z.* When the dual variates, $q \in \mathbb{R}^{S*}$, are prices or shadow prices, $\delta^*(q|z - Z)$ measures excess cost, foregone revenue, or foregone profit and is (minus) Nerlove's (1965) efficiency measure.[8] Because $\mathbb{R}^{S*}$ is the space of linear functionals on $\mathbb{R}^S$, $\delta^*(q|z - Z)$ measures the distance between the hyperplanes with normals $q$ that, respectively, include $z$ and that support $Z$. Regardless of interpretation, $\delta^*(\cdot|z - Z) \overset{*}{\longleftrightarrow} \delta(\cdot|z - Z)$. The Nerlove efficiency measure, as the support function for a translated convex set $z - Z$, is dual to $\delta(z|z - Z)$.

By definition, $\delta^*(q|z - Z) \leq 0$ for all $z \in Z$. We say that $z \in Z$ is *q-Nerlove inefficient* when $\delta^*(q|z - Z) < 0$. Because $(Z, \preceq_C)$ is a partially ordered set, one can encounter situations where a given $z$ is $\hat{q}$-Nerlove inefficient but not $q^o$-Nerlove inefficient for $\hat{q} \neq q^o$. To accommodate such outcomes, we have

**Definition 2.** *$z \in Z$ is Pareto inefficient if and only if $\delta^*(q|z - Z) < 0$ for all $q \in ri\ C^*$.*

If $z \in Z$ is Pareto inefficient, then

$$q'z < \delta^*(q|Z)$$

[8] Debreu (1951) uses the term 'dead loss'.

for all $q$ with $\delta^* (q|Z) + \delta^* (-q|Z) > 0$. Thus, Paretian inefficiency requires that $z \in ri\ Z$ (for example, Hiriart-Urruty and LeMaréchal 2001, Theorem C.2.2.3).

Dual methods, therefore, can distinguish between *inefficient* points lying inside $Z$ and its *efficient* boundary points. A well-known stumbling block to designing a dual programming algorithm to *measure* inefficiency is that dual variates, $q$, are determined only up to multiplication by a positive scalar. Hence, as the conjugacy between the support and indicator functions exemplifies, optimization in dual space can yield unboundedly large solutions that manifest themselves as infeasibilities in computational settings.

Traditional solutions in the inefficiency-measurement literature include restricting $q$ to the level set for a linear function of $q$ or to its associated closed half space. Debreu (1951, p.284) suggests "... dividing by a price index" and chooses the dual value of $z$ as the numeraire. Charnes et al. (1978) require that a subvector of $z$, which they term inputs, have a dual value of 1. Luenberger (1992) requires that the dual value of a predetermined element of $\mathbb{R}^S$ be at least 1 ensuring that the numeraire bundle remains constant. Ray (2007) extended the Luenberger approach by requiring that the dual value of two subvectors of $z$ corresponding to the inputs and outputs, respectively, at least equal 1. Pastor et al. (2012) generalize Ray (2007) by treating the case of an arbitrary number of linear inequalities.

The Minimum Norm Duality Theorem (Nirenberg 1961, Luenberger 1969, Theorem 5.13.1), albeit implicitly, defines another approach to normalizing $q$. It shows that the minimal distance between a point and the boundary of a convex set for a given norm is the maximal difference between the hyperplane through that point and the set's support function with dual variates restricted to fall within the unit disc defined by the dual norm.

We incorporate the inefficiency-measurement literature approaches and the minimum-norm approach under a common rubric and require that $q$ belong to a closed convex nonempty $Q \subset \mathbb{R}^{S*}$. Our *measure of Pareto Inefficiency*, $\delta^Q : \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined relative to $Q$ as:

$$\text{(11)} \qquad \begin{aligned} \delta^Q (z|Z) &\equiv \sup \{q'z - \delta^* (q|Z) : q \in Q\} \\ &= \sup_{q \in \mathbb{R}^{S*}} \{q'z - \delta^* (q|Z) - \delta (q|Q)\}. \end{aligned}$$

$\delta^Q (z|Z)$ isolates the element(s) of $Q$ for which the normalized $q$-Nerlove inefficiency is "...as nearly efficient as possible" (Afriat 1972). The superscript notation reminds us that $\delta^Q$ also has an interpretation as a "restricted" indicator function.

We choose (11) as the criterion function because of its longstanding importance in inefficiency measurement, its connection to minimum-norm problems, and its mathematical links to the conjugacy correspondence $\delta^* \overset{*}{\longleftrightarrow} \delta$ (for example, Debreu 1951, Nirenberg 1961, Luenberger 1969, Afriat 1972, Charnes et al. 1978, Ray 2004, 2007, Pastor et al. 2012, among others). Nevertheless, other writers use different criteria to induce inefficiency measures that include some of the measures induced below. For example, some impose stronger domain restrictions than ours and seek measures that maximize, for example, the ratio of revenue to cost, the ratio of realized revenue to maximal revenue, and the ratio of minimal cost to realized cost.[9] Others use prespecified notions of similarity or closeness (for example, Pastor, Ruiz, and Sirvent 2007). Färe, He, Li, and Zelenyuk (2019) study an approach to inefficiency measurement that relies on choosing netputs and choice variates $\lambda \in \mathbb{R}^N$ and $\theta \in \mathbb{R}^M$ to maximize a generic objective function $f(\lambda;\theta)$ for fixed prices subject to profitability constraints.[10]

## A Conjugacy Result

Lemma 1 and (11) give:

**Proposition 2.** $\delta^Q : \mathbb{R}^S \to \bar{\mathbb{R}}$ *is proper closed convex. Moreover,*

$$\delta^Q(z|Z) \overset{*}{\longleftrightarrow} \delta^*(q|Z) + \delta(q|Q) \tag{12}$$

$$\delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q) \geq q'z \quad \forall q, z \tag{13}$$

$$\hat{q} \in \partial\delta^Q(\hat{z}|Z) \Leftrightarrow \hat{z} \in \partial\delta^*(\hat{q}|Z) + \partial\delta(\hat{q}|Q) \Leftrightarrow \delta^Q(\hat{z}|Z) + \delta^*(\hat{q}|Z) + \delta(\hat{q}|Q) = \hat{q}'\hat{z}, \tag{14}$$

By construction, $\delta^Q(z|Z)$ is the conjugate of $\delta^*(q|Z) + \delta(q|Q)$ ensuring that it is a closed convex function of $z$. On the other hand, $\delta^*(q|Z) + \delta(q|Q)$ is proper closed convex which

[9] Note that it is routine in such settings to follow Charnes et al. (1978) and Tone (2001) and convert the resulting fractional programs into linear programs by setting the denominator of the fractional objective function to one. Such normalizations are special cases of $Q$.

[10] Färe et al. (2019) focus their attention on the case where N=1 and M=1.

with Lemma 1 establishes that $\delta^Q$ is proper and that $\delta^*(q|Z) + \delta(q|Q) \overset{*}{\longleftrightarrow} \delta^Q(z|Z)$. The remainder of the proposition also follows from Lemma 1.

Proposition 2 establishes a conjugacy correspondence between a mapping, $\delta^Q$, defined on $\mathbb{R}^S$ and one defined on its dual space $\mathbb{R}^{S*}$ that provides a description for how the choice of $Q$ affects $\delta^Q$. Following Debreu (1951), Charnes et al. (1978), Ray (2007), and others we have cast inefficiency measurement as a problem of choosing dual variates, $q \in Q$. But just as primal and dual algorithms exist for linear programs, we can reformulate (11) as optimizing over $\mathbb{R}^S$. The well-known conjugacy between the operations of *addition* and *infimal convolution* gives

$$\begin{aligned} \delta^Q(z|Z) &= \left(\delta^*(q|Z) + \delta(q|Q)\right)^*(z) \\ &= \underset{z=z^o+d}{inf}\left\{\delta(z^o|Z) + \delta^*(d|Q)\right\} \\ &= \inf_{d\in\mathbb{R}^S}\left\{\delta^*(d|Q) : z - d \in Z\right\} \end{aligned} \tag{15}$$

where $\underset{z=z^o+d}{inf}\left\{\delta(z^o|Z) + \delta^*(d|Q)\right\}$ defines the *infimal convolution* of $\delta$ and $\delta^*$ (for example, Rockafellar 1970 Theorem 16.4, Hiriart-Urruty 2001 Theorem E.3.2.1).[11]

Constructing the inefffiency measure, thus, reduces to locating the "minimal translation" of $z$ that maintains it as an element of $Z$, where the degree of minimality is measured by the support function for $Q$. The conjugate correspondence established, therefore, manifests a "minimal-support duality" that involves both $Z$ and $Q$. Geometrically, we translate $z$ until a tangency occurs between the boundaries of $Z$ and a level set for $\delta^*(d|Q)$. We reformulate (14) in equivalent terms as

$$q \in \partial\delta(z-d|Z) \cap \partial\delta^*(d|Q) \Leftrightarrow q \in \delta^Q(z|Z) \Leftrightarrow \delta^Q(z|Z) = \delta(z-d|Z) + \delta^*(d|Q)\,. \tag{16}$$

The conjugate manifestations of $\delta^Q$ as a (normalized) minimal difference between $q'z$ and $\delta^*(q|Z)$ and the infimal convolution of $\delta(\cdot|Z)$ and $\delta^*(\cdot|Q)$ mirror the optimization principles behind Luenberger's (1992) demonstration that benefit and shortage functions support calculation of Paretian efficient outcomes for a competitive market. Where Luenberger's demonstration reaffirms that Paretian-efficient calculations reflect assumptions on preferences and the technology, Proposition 2 shows that "Paretian inefficiency measurement"

[11] Some authors call the infimal convolution operation *epi-addition*. See, for example, Rockafellar and Wets 2009.

reflects assumptions on $Z$ and the numeraire embedded in the choice of $Q$. $\delta^Q$ is, in essence, a "joint product" of $Z$ and $Q$. The far left-hand side of (16),

$$q \in \partial\delta\left(z - d|Z\right) \cap \partial\delta^*\left(d|Q\right),$$

which requires that the subdifferentials of $\delta\left(z^o|Z\right)$ and $\delta^*\left(z - z^o|Q\right)$ overlap (tangencies between boundaries), manifests the *Dubovitskii-Milyutin Lemma* that characterizes extremals set-valued optimization problems (Isac and Khan 2008).

Different formulations of (11) can yield different perspectives on the same convex optimization problem. Those different perspectives often give different naming conventions. For example, $d$ variates in (15) can be *slacks*, but they also can be Lagrange multipliers depending upon the perspective. Because they depict the *directions* in $\mathbb{R}^S$ in which $z$ is projected onto $EffZ$, following Ray (2007) we call them *directions* and the $\bar{z} = z - d$ variates, *projections*. We define:

$$D^Q\left(z|Z\right) \equiv \left\{d : q \in \delta^*\left(d|Q\right) \quad \forall q \in \partial\delta^Q\left(z|Z\right)\right\} \tag{17}$$

as the *endogenous directions* for (11) and

$$P^Q\left(z|Z\right) \equiv \left\{z - d : d \in D^Q\left(z|Z\right)\right\}, \tag{18}$$

as the *endogenous projections.*

We close this discussion with basic results on $D^Q\left(z|Z\right)$ and the ability of $\delta^Q\left(z|Z\right)$ to determine whether $z$ falls within $Z$.

**Lemma 3.** *Let (11) have a finite solution. Then* $D^Q\left(z|Z\right) \subset Q^*_\infty$ *for all* $z$.

Proof: See Appendix.

**Proposition 3.** *Let (11) have a finite solution.*

*a)* $z \in Z \Rightarrow \delta^Q\left(z|Z\right) \leq 0$*;*

*b) if* $Q^*_\infty \subset C$ *then* $\delta^Q\left(z|Z\right) \leq 0 \Rightarrow z \in Z$*; and*

*c) if* $Q$ *is bounded,* $\delta^*\left(q|z - Z\right) > 0 \Rightarrow z \notin Z$.

Proof: See Appendix.

Färe and Lovell (1978) showed the formal relationship between Farrell's (1957) inefficiency score and Shephard's (1953, 1970) distance function establishing that inefficiency measures can be cardinal (function) representations of sets. From the time that Minkowski (1911) introduced his *Distanzfunktion*, Minkowski functionals, gauge functions, and co-gauge (distance) functions have played a key role in characterizing star-shaped and convex sets (Aliprantis and Border 2007). And because these functionals are all based on some measure of a point's distance from a set's boundary, their interpretation as an inefficiency function is natural.[12] It is well-known, however, that not all inefficiency measures provide function representations of their associated $Z$. Lemma 3 and Proposition 3 detail restrictions on $C$ and $Q$ that affect a chosen measure's ability to represent $Z$ exhaustively.

## Antecedents

Propositions 2 and 3 have antecedents in both the broader optimization literature and in the narrower inefficiency-measurement literature. Nirenberg's *Minimum-Norm Duality Theorem* (Nirenberg 1961 and Luenberger 1969) is of particular note. As we discuss below, the support and the gauge functions for a closed convex set are polar to one other. And gauge functions for compact zero-symmetrical sets on $\mathbb{R}^S$, in turn, form a one-to-one correspondence with their norms (see, for example, Rockafellar 1970 Section 15). Hence, (15) encompasses minimum-norm measures as special cases. Briec (1997), Briec and Lesourd (1999), and Petersen (2018) examine related measures.

Different studies show that changing the price normalization changes the resulting measure. For example, early authors recognized that normalizing the dual value of "inputs" and normalizing the dual value of "outputs" gave different inefficiency measures. Following Luenberger (1992), Chambers, Chung, and Färe (1996, 1998) generalized that observation to distinguish between radial input and output measures, input-directional distance functions, output-directional distance functions, and technology-directional distance functions. Ray (2007) generalized further by normalizing input and output vectors separately (also see Aparicio, Pastor, and Ray 2013). Cooper, Pastor, Aparicio, and Borras (2011) relate the

[12] Indeed, Newman (1987) introduces his survey of gauge functions in economics by calling them "...sensible measure(s) of efficiency".

Russell efficiency measure to (minus) the $l^{\infty}$ norm. Pastor et al. (2012) show that minimizing $\delta^{*}(q|z-Z)$ subject to different normalizations yields different inefficiency measures. And by restricting attention to the canonical DEA model and normalization conditions "...represented by means of a finite set of equalities and/or inequalities...", they induce versions of the Banker et al. (1984) measure, a directional distance function, the weighted-additive measure, and the Russell measure.[13] Färe, Grosskopf, and Whitaker (2013) derive an *endogenous directional measure*. Aparicio, Borras, Pastor, and Vidal (2015) show that the Russell efficiency measure is conjugate to the revenue function subject to dual variates falling in $Q$ that is a special case of Proposition 4.h below.

## Conjugate Correspondences

Our analysis starts with $Z$, isolates $EffZ$, and then uses (11) to measure inefficiency. Proposition 2 implies that an equivalent, equally relevant, dual approach exists. One can start with a proper closed convex inefficiency measure, $\delta^{Q}(z|Z)$, and then use the conjugacy correspondence to resurrect a conjugate $\delta^{*}(q|Z)+\delta(q|Q)$ that is proper closed and convex. Proposition 2 ensures the induced $\delta^{*}(q|Z)+\delta(q|Q)'s$ consistency with $\delta^{Q}(z|Z)$ without the need for "...difficult constructive arguments" (McFadden 1978). Broad classes of functions are closed convex. By Proposition 2, each such class defines a class of inefficiency measures and a conjugate dual class of $\delta^{*}(q|Z)+\delta(q|Q)'s$.

**Example 4.** *Let* $\delta^{Q}(z|Z)=\sum_{s} z_s$. *Then*

$$\delta^{*}(q|Z)+\delta(q|Q)=a+\begin{cases}0 & \text{if } q=1\\ \infty & \text{otherwise}\end{cases}$$

**Example 5.** *Let* $\delta^{Q}(z|Z)=||z||-a$. *Then*

$$\delta^{*}(q|Z)+\delta(q|Q)=a+\begin{cases}0 & \text{if } ||q||_{*}\leq 1\\ \infty & \text{otherwise}\end{cases}$$

[13] Because Pastor et al. (2012) do not impose convexity on their normalizing set in their general model, it is not a special case of ours. But the listed representation results are all developed for the polyhedral case which is covered by Proposition 2.

*where*

$$||q||_* \equiv \sup_x \{q'z : ||z|| \leq 1\}.$$

**Example 6.** *Let* $\delta^Q(z|Z) = a - \max_{\mu,\lambda\in\Delta_J}\left\{\mu : \mu z \leq \sum_j \lambda_j z^j\right\}$ *where* $\Delta_J \equiv \left\{x \in \mathbb{R}_+^S : \sum_s x_s = 1\right\}$ *denotes the unit simplex in* $\mathbb{R}^S$ *and* $z^j \in \mathbb{R}^S,\ j = 1,\ldots,J$. *Then*

$$\delta^*(q|Z) + \delta(q|Q) = \sup_{z,\mu,\lambda\in\Delta_J}\left\{q'z + \mu : \mu z \leq \sum_j \lambda_j z^j\right\} - a$$

**Example 7.** *Let* $\delta^Q(z|Z) = \delta^*(z|X^*) - a$ *for nonempty closed convex* $X^* \in \mathbb{R}^{S*}$. *Then* $\delta^*(q|Z) + \delta(q|Q) = a + \delta(q|X^*)$.

**Example 8.** *Let Let* $\delta^Q(z|Z) = \delta(z|X) - a$ *for nonempty closed convex* $X \in \mathbb{R}^S$. *Then* $\delta^*(q|Z) + \delta(q|Q) = a + \delta^*(q|X)$

**Example 9.** *Let* $f^n : \mathbb{R}^S \to \bar{\mathbb{R}},\ n = 1,\ldots,N$ *be proper closed convex and*

$$\delta^Q(z|Z) = \max\left\{f^1,\ldots,f^N\right\}.$$

*Then* $\delta^Q(z|Z)$ *is closed convex (for example, Rockafellar (1970, Theorem 5.5)) and*

$$\begin{aligned}
\delta^*(q|Z) + \delta(q|Q) &= \sup_z \left\{q'z - \max\left\{f^1(z),\ldots,f^N\{z\}\right\}\right\} \\
&= \sup_z \min_n \{q'z - f^n(z)\} \\
&= \min_n \{f^{n*}(q)\}
\end{aligned}$$

## Composition Results

As Examples 4-9 illustrate, constructing the conjugates for simple choices of $\delta^Q$ is straightforward. The same is true for simple choices of $Z$ and $Q$. But practical instances may require more complex settings and more complex manipulations. In such instances, one strategy is to follow McFadden (1978) and solve parts of the conjugacy correspondence for which the underlying dual relationships are tractable and then use Proposition 2. The next proposition presents composition rules for some common convex forms.

**Proposition 4.** *a) Let $f^n : \mathbb{R}^S \to \bar{\mathbb{R}},\ n = 1, \ldots, N$ be proper closed convex and*

$$\delta^Q(z|Z) = \sum_n f^n(z).$$

*Then*

$$\delta^*(q|Z) + \delta(q|Q) = \inf_{q^1,\ldots,q^N} \left\{ \sum_n f^{n*}(q^n) : \sum_n q^n = q,\ q^n \in \mathbb{R}^{S*}, n = 1, \ldots, N \right\}$$

*b) Let $f^n : \mathbb{R}^S \to \bar{\mathbb{R}},\ n = 1, \ldots, N$ be proper closed convex and*

$$\delta^Q(z|Z) = \inf_{z^1,\ldots,z^N} \left\{ \sum_n f^n(z^n) : \sum_n z^n = z,\ z^n \in \mathbb{R}^S, n = 1, \ldots, N \right\}.$$

*Then*

$$\delta^*(q|Z) + \delta(q|Q) = \sum_n f^{n*}(q)$$

*c) Let $\delta^Q(z|Z) \equiv \max_{j=1,\ldots,J} \{z'\bar{q}^j - b_j\}$ with $\bar{q}^j \in C^*$ and $b_j \in \mathbb{R}$ $j = 1, \ldots, J$. Then (co $\{\cdot\}$ denotes the convex hull of $\{\cdot\}$ in the following)*

$$\delta^*(q|Z) + \delta(q|Q) = \min_{\lambda} \left\{ \sum_{j=1}^{J} \lambda_j b_j : \lambda \in \Delta_J \ \ q = \sum_{j=1}^{J} \lambda_j \bar{q}^j \right\},$$

*for all $q \in co\left\{\bar{q}^1, \ldots, \bar{q}^J\right\} = dom\ \delta^{Q*}$*

*d) Let $z^j \in \mathbb{R}^S,\ j = 1, \ldots, J$ and*

$$\delta^*(q|Z) = \begin{cases} \max_{j=1,\ldots,J} \{q'z^j\} & \text{if } q \in C^* \\ \infty & \text{otherwise.} \end{cases}$$

*Then for $z \in dom\ \delta^Q$:*

$$\delta^Q(z|Z) = \inf_{d,\lambda} \left\{ \delta^*(d|Q) : \lambda \in \Delta_J, z - d \in \sum_{j=1}^{J} \lambda_j z^j + C \right\}$$

*e) Let $Z = C$. Then*

$$\delta^Q(z|Z) = \inf_{z^o} \{\delta^*(z - z^o|Q) : z^o \in C\}$$

*f) Let $Z = Z^1 \cap Z^2 \cap \cdots \cap Z^K$ with each $Z^k \subset \mathbb{R}^S$ nonempty closed convex and $\bigcap_k ri\ Z^k \neq \emptyset$. Then*

$$\delta^Q(z|Z) = \sup_{q^1,\ldots,q^K} \left\{ \sum_k \delta^*\left(q^k|z - Z^k\right) - \delta\left(\sum_k q^k|Q\right) \right\}$$

*g) Let* $\delta^*(q|Z) = \sum_k \delta^*(q|Z^k)$ *with* $Z^k \in \mathbb{R}^S,\ k = 1\dots,K$ *closed convex. Then*

$$\delta^Q(z|Z) = \inf_{z^1,\dots,z^K}\left\{\sum \delta\left(z^k|Z^k\right) + \delta^*\left(z - \sum_k z^k|Q\right)\right\}.$$

*h) Let* $Q = Q^1 \cap Q^2 \cap \dots \cap Q^K$ *with each* $Q^k \subset \mathbb{R}^{S*}$ *nonempty closed convex and* $\bigcap_k ri\ Q^k \neq \emptyset$*. Then*

$$\delta^Q(z|Z) = \inf_{d^1,\dots,d^K}\left\{\delta\left(z - \sum_k d^k|Z\right) + \sum_k \delta^*\left(d^k|Q^k\right)\right\}$$

.

*i) Let* $\delta^*(d|Q) = \sum_k \delta^*(d|Q^k)$ *with* $Q^k \in \mathbb{R}^{S*},\ k = 1\dots,K$ *closed convex. Then*

$$\delta^Q(z|Z) = \inf_d\left\{\delta(z - d|Z) + \sum_k \delta^*\left(d|Q^k\right)\right\}.$$

*j) Let* $q^j \in \mathbb{R}^{S*},\ j = 1,\dots,J$ *and*

$$\delta^*(d|Q) = \begin{cases} \max_{j=1,\dots,J}\{d'q^j\} & \textit{if } d \in Q^*_\infty \\ \infty & \textit{otherwise.} \end{cases}$$

*Then for* $z \in dom\ \delta^Q$

$$\delta^Q(z|Z) = \sup_{q,\lambda\in\Delta_J}\left\{\delta^*(q|z - Z) : q \in \sum_{j=1}^J \lambda_j q^j + Q_\infty\right\}.$$

*k) Let* $Q = K^*$ *with* $K^* \subset C^*$ *a nonempty closed convex cone.*

$$\delta^Q(z|Z) \ = \ \sup\{\delta^*(q|z - Z) : q \in K^*\}.$$

*l) Let* $Q = \{\tilde{q}\}$*. Then*

$$\delta^Q(z|Z) = \delta^*(\tilde{q}|z - Z).$$

*m) Let* $0 \in Q$*. Then*

$$\delta^Q(z|Z) = \inf_d\{\delta(z - d|Z) + \gamma(d|Q_*)\}$$

*with* $Q_* \equiv \left\{x \in \mathbb{R}^S : \delta^*(x|Q) \leq 1\right\} \mathbb{R}^S$ *closed convex.*

Proof: See Appendix.

## Discussion of Composition Rules

Antecedents exist for a number of the results in Proposition 4. We try to highlight them in the following discussion. Note, first, that different restrictions on $Z$ and $Q$ can yield the same $\delta^Q$. The jointness inherent in $\delta^*(q|Z) + \delta(q|Q)$ can manifest itself in identification issues associated with isolating the precise structures that generate $\delta^Q$. Further evidence of the inherent identification problem comes from the Färe et al. (2019) demonstration of alternative strategies for generating Russell-type measures.

Parts a) through c) of Proposition 4 develop a set of calculus rules for the conjugacy operation $\delta^Q(z|Z) \overset{*}{\to} \delta^*(q|Z) + \delta(q|Q)$ for some familiar convex forms. Cases a) and b) correspond to addition and infimal convolution.

Case c) uses the observation that a closed convex function is the pointwise supremum of the affine functions that it majorizes to construct a closed convex function on $\mathbb{R}^S$ from elements of $\mathbb{R}^{S*}$ and $\mathbb{R}$. It supports computation of an inefficiency measure in instances where prior knowledge or observation contain information on dual variates $q$. For example, let $(\bar{z}^j, \bar{q}^j) \in \mathbb{R}^S \times \mathbb{R}^{S*}$, $j = 1, \ldots, J$ represent $J$ observations on $z$ and $q$. Then setting $b_j = \bar{q}^{j\prime}\bar{z}^j$ gives

$$\delta^Q(z|Z) = \max_{j=1,\ldots,J} \left\{ \bar{q}^{j\prime}\left(z - \bar{z}^j\right) \right\}$$

and

$$\delta^*(q|Z) + \delta(q|Q) = \min_{\lambda} \left\{ \sum_{j=1}^{J} \lambda_j \bar{q}^{j\prime}\bar{z}^j : \lambda \in \Delta_J \;\; q = \sum_{j=1}^{J} \lambda_j \bar{q}^j \right\},$$

as the dual conjugates (Hanoch and Rothschild, 1972). Both are computable using linear programming techniques. Here *dom* $\delta^*(q|Z) + \delta(q|Q) = co\left\{\bar{q}^1, \ldots, \bar{q}^J\right\}$.

Parts d)-m) give calculus rules for the conjugacy operation $\delta^Q(z|Z) \overset{*}{\leftarrow} \delta^*(q|Z) + \delta(q|Q)$ for different $Z$ and $Q$.

Case d) is the most familiar. It generalizes the Banker et al. (1984) variable-returns model to accommodate netputs and a general $Z_\infty$.[14] The resulting conic program isolates endogenous projections of $z$ falling in $co\left\{z^1, \ldots, z^J\right\} + C$ that support the boundary of $Q$.[15]

[14]The related literature using the canonical DEA model is vast and impractial to cite parsimoniously.

[15]One computes the Charnes et al. (1978) and Banker et al. (1984) measures using linear programming techniques. The generalization requires conic programming (Boyd and Vandenberghe 2004, Nemirovski 2007,

When coupled with d), different choices for $Q$ generate different classes of existing inefficiency measures.

As a first example, let $Q = \{\tilde{q}\}$ with $\tilde{q} \in C^*$. Then

$$\begin{aligned}\delta^Q(z|Z) &= \min_{\lambda}\left\{\tilde{q}\left(z - \sum_j \lambda_j z^j\right) : \lambda \in \Delta_J\right\} \\ &= \delta^*\left(\tilde{q}|z - co\left\{z^1, \dots, z^J\right\}\right),\end{aligned}$$

which is the $\tilde{q}$-Nerlovian inefficiency for $z$ for $co\left\{z^1, \dots, z^J\right\}$. Special cases include netput versions of Färe and Lovell's (1978) *Russell* efficiency measure, the Charnes et al. (1985) additive efficiency measure, and the weighted-additive efficiency measure (Lovell and Pastor 1995, Cooper et al. 2011, Aparicio, Pastor, and Vidal 2016, Chambers 2023).[16] Each reduces to a support function for the Minkowski set difference $z - Z$ which is the dual conjugate of the indicator function for that set difference.

Now let $Q = \{q \in C^* : q'z = 1\}$. Then

$$\begin{aligned}\delta^Q(z|Z) &= \inf_{\lambda}\left\{\sup_q\left\{q'\left(z - \sum_j \lambda_j z^j\right) : q'z = 1\right\} : \lambda \in \Delta_J\right\} \\ &= \inf_{\lambda}\sup_q\left\{q'\left(z - \sum_j \lambda_j z^j\right) : q'z = 1, \lambda \in \Delta_J\right\} \\ &= \sup_q\left\{q'z - \delta^*\left(q|co\left\{z^1, ..., z^J\right\}\right) : q'z = 1\right\}\end{aligned}$$

where the third equality follows by the Saddlepoint Theorem. The result is a transformation of a netput version of the Farrell (1957) inefficiency measure.

Applied inefficiency analysts often segregate inputs and outputs. Now partition $z$ as $z' = (\tilde{z}', \hat{z}')$ with $\tilde{z}' \in \mathbb{R}^M$ and $\hat{z} \in \mathbb{R}^{S-M}$, partition $q$ conformably, and define $Q = \{q \in C^* : \tilde{q}'\tilde{z} = 1\}$. Then

$$\delta^Q(z|Z) = 1 + \sup_{q \in Z^*_\infty}\left\{\hat{q}'\hat{z} - \delta^*\left(q|co\left\{z^1, \dots, z^J\right\}\right) : \tilde{q}'\tilde{z} = 1\right\},$$

defines a transformation of the generalization of the Charnes et al. (1978) and Banker et al. (1984) inefficiency measures that accommodates segregating inputs and outputs and

Bertsekas 2016).

[16] Ray (2004) shows the structural similiarity of the additive, weighted-additive, and Russell measures.

other partitionings of netputs needed to accommodate the presence of quasi-fixed vs.variable inputs, desirable, undesirable outputs, and other departures from the canonical set up.

The example of conical $Z$, case e), crystallizes the role that $Q$ plays in determining $\delta^Q(z|Z)$. Figure 1 illustrates. There, the left-hand panel illustrates conical $Z$ and the right-hand panel *dom* $\delta^*(q|Z)$. Constant returns ensures that

$$\delta^*(q|Z) = \begin{cases} 0 & \text{if } q \in Z_\infty^* \\ \infty & \text{otherwise.} \end{cases}$$

For the depicted $z \in Z$, $q'z \leq 0$ for all $q \in Z_\infty^*$, thus

$$\sup\{q'z - \delta^*(q|Z) : q \in Z_\infty^*\} = 0.$$

Let $Q = \{q \in Z_\infty^* | q_1 z_1 = 1\}$, depicted by the closed line segment connecting $q$ and $\left(\frac{1}{z_1}, 0\right)$. Then $\delta^Q(z|Z) = \hat{q}'z < 0$, where the inequality is confirmed by noting that the angle formed by $z$ and $\hat{q}$ is obtuse.

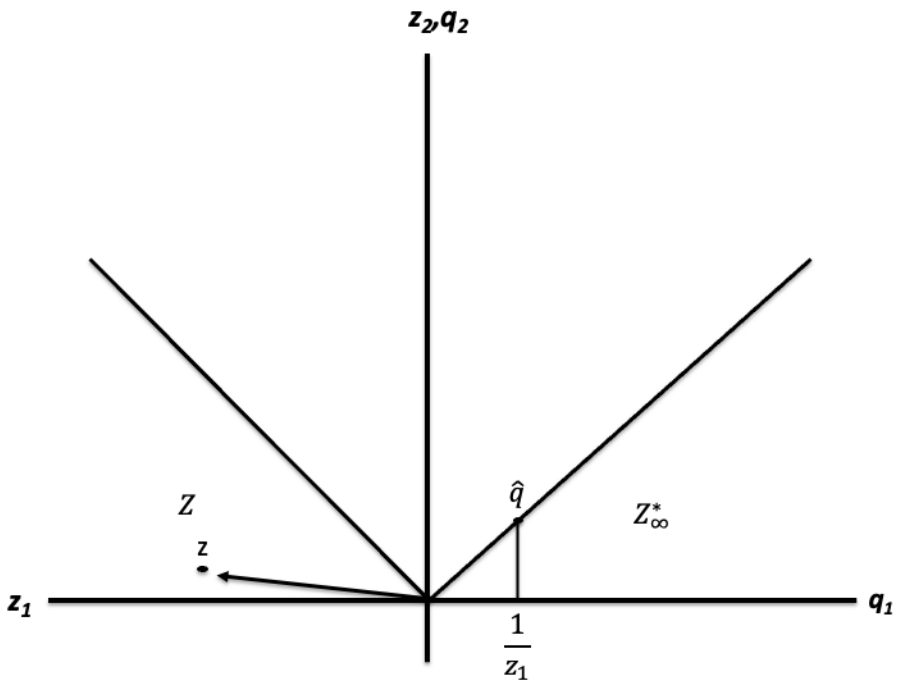
Cases f) and g) characterize $Z$ exhibiting different forms of decomposability or separability. Taking each $Q^k$ to be a closed half space gives polyhedral $Z$ as a special case. Therefore, case d) is a special case of f). Case f) also includes what Frisch (1965) calls *multi-dimensional assortment*, where $Z$ is composed of subsets of $\mathbb{R}^S$ that represent sub-processes or stages in production. Such structures have played an important role in extending canonical DEA models to accommodate joint production of desirable and undesirable outputs (for example, Ayres and Kneese 1969, Førsund 1998, Coelli, Lauwers, and Huylenbroeck 2007, Podinovski and Kuosmanen 2011, Murty, Russell, and Levkoff 2012, Chen and Delmas 2012, Chambers, Serra, and Lansink 2014, Murty and Russell 2022).

To illustrate, partition $z$ as $z = (z_d, z_u, z_i)'$ and let $Z = Z^D \cap Z^U$ where

(19)
$$\begin{aligned} Z^D &= \left\{z \in \mathbb{R}^S : z_i \textit{ can produce } z_d\right\}, \textit{ and} \\ Z^U &= \left\{z \in \mathbb{R}^S : z_i \textit{ can produce } z_u\right\}. \end{aligned}$$

Here subscript $d$ denotes desirable outputs, $u$ denotes undesirable outputs or byproducts, and $i$ denotes inputs. $Z^D$ represents the desirable production process and $Z^U$ the clean-up or abatement process. Kohli (1983) classifies the form in (19) as “output-price nonjoint”.

Figure 1: Figure 1: Constant Returns and Q



It depicts a production process that "cracks" a fixed input-bundle, $z_i$, into separate bundles of desirable and undesirable outputs. The output-price nonjoint specification is common in pollution-abatement studies, where desirable and undesirable outputs are often assumed to be produced in fixed proportions. Then

$$\delta^Q(z|Z) = \sup_{q^U,q^D}\left\{\delta^*\left(q^D|z - Z^D\right) + \delta^*\left(q^U|z - Z^U\right) - \delta\left(q^D + q^U|Q\right)\right\}.$$

By (14)

$$q^U + q^D \in \partial\delta^Q(z|Z) \Leftrightarrow z \in \partial\delta^*\left(q^U|Z^U\right) \cap \partial\delta^*\left(q^D|Z^D\right) \cap \partial\delta\left(q^U + q^D|Q\right),$$

which shows that the "virtual price" of $z$ splits into two components, one for $Z^D$ and one for $Z^U$.

Case f) also includes the event-contingent technologies (Chambers and Quiggin 2000, O'Donnell and Griffiths 2006, Chambers, Hailu, and Quiggin 2011, Chambers, Serra, and Stefanou 2015, Serra, Chambers, and Lansink 2014, Chambers et al. 2014) used in studies of inefficiency in the presence of uncertainty.

Case g) models $Z$ as the infimal convolution of $J$ distinct sub-processes

$$\delta(z|Z) = \inf_{z^1,\ldots,z^J}\left\{\sum_j \delta\left(z^j|Z^j\right) : \sum_j z^j = z\right\}$$

It characterizes, for example, decisionmakers or enterprises that operate across separate plants or locations. It also generalizes familiar notions of input and output nonjoint technologies to netput-nonjointness that allows netputs to be freely allocated across the sub-processes.

Cases h), i), and j) model situations where the normalization associated with (11) can be decomposed into subsets of normalizing restrictions. Case h) treats $Q$ as the intersection of a finite series of convex sets. Restricting each $Q^k$ to be polyhedral, cases h) and j) generalize broad classes of DEA-based measures including Charnes et al. (1978), Banker et al. (1984), directional distance functions, weighted average measures, various Russell measures, Ray (2007), and the DEA measures studied by Pastor et al. (2012).[17]

[17] As noted the weighted average and Russell measures are also special cases of $Q = \{\tilde{q}\}$.

To illustrate, let $Q = C^* \cap Q^2$ with

$$Q^2 = \left\{ (\tilde{q}', \hat{q}') \in \mathbb{R}^{S*} : \tilde{q}_m = \frac{1}{M z_m}, \hat{q}_n = \frac{\tau z_S}{(S-M) z_n}, \tau \in \mathbb{R},\ m = 1, \ldots, M, n = M+1, \ldots, S \right\},$$

Then

$$\delta^Q (z|Z) = \sup_{\tau} \left\{ 1 + \tau - \delta^* (q|Z) : q \in Z^*_\infty \cap Q^2 \right\},$$

which is a transformation of the generalization of the Pareto-Koopmans inefficiency measure to arbitrary $Z$ and $Z_\infty$ (see, for example, Ray 2004, expression (5.19))

Now let $Q = Q^1 \cap Q^2 \cap Z^*_\infty$ with $Q^1 = \{q : q'g^1 \leq 1\}$ and $Q^2 = \{q : q'g^2 \leq 1\}$ with $g^1, g^2 \in \mathbb{R}^S$. Then

$$\delta^Q (z|Z) = \sup_{q \in Z^*_\infty} \left\{ q'z - \delta^* (q|Z) : q'g^1 \leq 1, q'g^2 \leq 1 \right\},$$

which generalizes Ray's (2007) Overall (Shadow Profit) inefficiency measure (Ray 2007, Aparicio et al. 2013, and Ray and Yang 2024) to accommodate arbitrary $Z$ and $Z_\infty$

Cases k) and l) are straightforward consequences of (11) and need little elaboration.

Case m) shows that, when the origin belongs to $Q$, the class of finite Paretian inefficiency measures specializes to the class of "minimal gauge functions"

$$\delta^Q (z|Z) = \inf_d \{ \gamma (d|Q_*) : z - d \in Z \}.$$

The class of "minimal-norm inefficiency measures", in turn, is the subset of the minimal-gauge inefficiency measures for bounded $Q$ symmetric about zero. A *norm*, $\rho : \mathbb{R}^S \to \bar{\mathbb{R}}$, satisfies: a) $\rho(x) \geq 0$ for all $x \in \mathbb{R}^S$, b) $\rho(\alpha x) = |\alpha| \rho(x)$ for $\alpha \in \mathbb{R}$; and c) subadditivity (the triangle inequality). Let $Q$ be closed convex bounded and symmetric about 0. Then $\gamma(q|Q)$ defines a norm for $\mathbb{R}^{S*}$, $\rho_* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, whose polar form

$$\begin{aligned} \rho(d) &= \sup_q \{ q'd : \rho_*(q) \leq 1 \} \\ &= \sup_q \{ q'd : \gamma(q|Q) \leq 1 \} \\ &= \delta^* (d|Q) \end{aligned}$$

is a norm for $\mathbb{R}^S$ with unit ball $\left\{ d \in \mathbb{R}^S : \delta^* (d|Q) \leq 1 \right\}$ (see, for example, Rockafellar 1970, Theorem 15.2). Special cases of $\rho$ include the $L_p$ norms used to define the Hölder distance functions studied by Briec (1997) and Briec and Lesourd (1999). Taking $\rho_*$ to be the Euclidean norm $||\cdot||$ so that $Q = \left\{q \in \mathbb{R}^{S*} : ||q|| \leq 1\right\}$ gives

$$\delta^Q (z|Z) = \inf_d \{||d|| : z - d \in Z\}$$

as the minimum Euclidean distance to translate $z$ while maintaining that it belongs to $Z$.

# Mind the Gap: Exogenous q and Measuring Dual Inefficiency

Oftentimes, exogenous information on dual variates, say $q^o$, as well as $z$ is available. Coupled with knowledge of $Z$, that allows calculation of the $q^o$-Nerlovian inefficiency for $z$. Because $\delta^Q (z|Z)$ maximizes Nerlovian inefficiency over $Q$, it can diverge from $\delta^* (q^o|z - Z)$. A tradition that traces to Farrell (1957) uses that observation to decompose $q^o$-Nerlovian efficiency for $z$ for exogenous $q^o$ into a *technical inefficiency* component and a *dual inefficiency* component.[18]

## Fenchel's Inequality

Fenchel's Inequality (3) offers a natural means to examine efficiency decompositions. Applying it to $\delta^Q$ gives the recycled version of (13):

$$\delta^Q (z|Z) \geq \delta^* (q|z - Z) - \delta (q|Q) \qquad (20)$$

for all $z, q$. Using (20), we define $\varphi^Q : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}_+$ as

$$\varphi^Q (q, z|Z) \equiv \begin{cases} \delta^Q (z|Q) + \delta^* (q|Z) - q'z & \text{if } q \in Q \\ \infty & \text{otherwise.} \end{cases}$$

[18] More common terms for dual inefficiency are either *allocative* or *price* inefficiency. For example, Afriat (1972, p. 582), who worked in a cost context, partitioned the "total gap...as a part due to the inefficiency of output with respect to the input, and...misplaced allocation of cost over inputs". When $\mathbb{R}^{S*}$ represents price, "virtual-price", or "shadow-price" space, the allocative or price inefficiency interpretation is natural. The dual terminology covers both and is more descriptive of its mathematical structure. Thus, it also applies to non-price settings.

$\varphi^Q$ is closed convex as a function of $z$, closed convex as a function of $q$, closes any duality gap implied by a strict inequality in (20), and *for* $q \in Q$ measures the difference between $z's$ Pareto inefficiency and its Nerlovian inefficiency. Thus,

$$\delta^* (q|z - Z) = \delta^Q (z|Z) - \varphi^Q (q, z|Z), \tag{21}$$

for $q \in Q$ and all $z$.

Expression (21) invites the interpretation of $\varphi^Q (q^o, z)$ as the *dual inefficiency* component of Nerlovian inefficiency at $q^o$ and of $\delta^Q (z|Z)$ as the *technical inefficiency.* Interpretive issues intrude, however, for $q^o \notin Q$ because $\varphi^Q (q^o, z|Z)$ then becomes arbitrarily large. That implies an infinitely large dual inefficiency conveying the intuition that relative to $q^o$ a decisionmaker at $z$ makes arbitrarily large bad decisions, when in truth $\delta^* (q^o|z - Z)$ and $\delta^Q (z|Z)$ are not commensurable. This intuition founders because the one-to-one relationship between $\delta^Q (z|Z)$ and $\delta^* (q|z - Z)$ becomes noninformative when $q \notin Q$.

Recent work on *axiomatic inefficiency measurement* provides another lens through which to analyze this aspect of $\varphi^Q$. The axiomatic approach evaluates different inefficiency measures by their ability to satisfy certain axioms (see, for example, Russell and Schworm 2009 and 2017). Usually, axioms are imposed on the proposed technical inefficiency measure, our $\delta^Q$. Aparicio, Zofio, and Pastor (2023) argue that technical inefficiency measures should also be judged on the behavior of their associated dual inefficiency measures. They propose that a candidate dual inefficiency measure, $I : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}$, satisfy an *Essential Property* summarized, in our notation, as[19]

**Property 1.** *If* $z \in \partial\delta^* (q|Z)$*, then* $I (q, z) = 0$.

Aparicio et al. (2023) use examples to show that the Russell Graph Measure, the Enhanced Russell Graph, the Additive, and the Weighted Additive measures do not satisfy Property 1. Because these measures correspond to cases rationalized by a singleton $Q = \{\tilde{q}\}$, $\varphi^Q (q, z|Z) = \infty$ for $q \neq \tilde{q}$ for them. Failure of a measure to satsify Property 1 translates in our setting into noncommensurability between Nerlovian and Paretian inefficiency.

[19] Aparicio et al. (2023) segregate inputs and outputs and discuss separate input and output-oriented versions of their Essential Property for both ratio-based and difference-based measures. We only consider difference-based measures for netputs and leave the natural extensions to the interested reader.

More generally, as a consequence of $\varphi^Q$'s convexity and Proposition 2, we have:

**Corollary 1.** *Let $\delta^Q(z|Z)$ be finite. Then $\varphi^Q$ is zero-minimal if and only if*

$$q \in \partial\delta^Q(z|Z) \Leftrightarrow z \in \partial\delta^*(q|Z) + \partial\delta(q|Q)$$

The Russell measures, the Additive measure, and the Weighted Additive cannot satisfy Corollary 1 for arbitrary $q$ when $Q = \{\tilde{q}\}$ because then $\partial\delta(q|Q) = \emptyset$ for $q \neq \tilde{q}$. Figure 2 illustrates the phenomenon. We assume that $Z_\infty = \mathbb{R}^2_-$ and the efficient set is the curve labelled $0E$. First, let $Q$ be the closed line segment connecting $(1,0)$ and $\hat{q}$. Then the set of efficient projections is restricted to points falling below $z^*$ on $OE$. In particular, points on the arc beyond $z^*$ on $OE$ cannot be efficient projections.

Now, let $Q = \{q : q'g = 1\}$ (a directional distance function). $Q$ now encompasses all directions in $C^*$ so that no point on $OE$ is excluded from $P^Q(q|Q)$. $\varphi^Q$ derived from these measures satisfies Property 1.

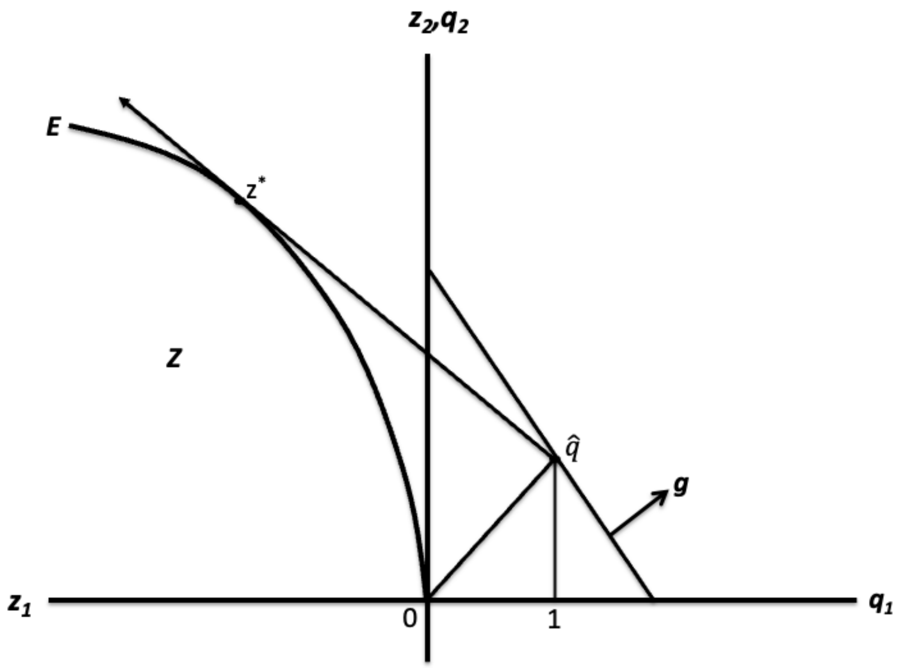
The technical issue is that certain choices for $Q$ can restrict the range of $P^Q(z|Z)$ enough to make the link between $\delta^Q(z|Z)$ and $\delta^*(q|z-Z)$ implied by Fenchel's Inequality noninformative. Or more simply, as Chambers (2023) observed, choosing a Nerlovian inefficiency measure for a fixed $\tilde{q}$ to measure technical inefficiency, when information on exogenous $q$ is available, distorts the distinction between technical and dual inefficiency.

We note that our use of Fenchel Inequality's, (20), differs from related discussions in some studies (see, for example, Färe and Grosskopf 1997, Chambers et al. 1998, Chambers and Färe 2004, Cooper et al. 2011, Zofio, Pastor, and Aparicio 2013, Aparicio, Borras, Pastor, Vidal 2013, Petersen 2018). The difference is more semantic than substantive, but it merits clarification. Ours follows established terminology in convex analysis, but some inefficiency studies use different naming conventions.

As a first example, let $Q = \left\{q \in \mathbb{R}^{S*} : q'g = 1\right\}$ with $g \in Z_\infty$. Then

$$\begin{aligned} \delta^Q(z|Z) &= \sup_q \{q'z - \delta^*(q|Z) - \delta(q|Q)\} \\ &= \sup_q \{q'z - \delta^*(q|Z) : q'g = 1\} \\ &= \sup_q \left\{\frac{q}{q'g}{}' z - \delta^*\left(\frac{q}{q'g}|Z\right)\right\}, \end{aligned} \tag{22}$$

Figure 2: Figure 2: Essential Property Fails



where we use the sublinearity of $\delta^*$. Consequently, reasoning similar to (3) gives:

$$\delta^Q(z|Z) \geq \frac{q}{q'g}'z - \delta^*\left(\frac{q}{q'g}|Z\right) \text{ for all } q, z. \tag{23}$$

Different writers have named (23) and its analogues differently, including the *Luenberger Inequality*, the *Fenchel-Mähler Inequality*, the *generalized Fenchel-Young Inequality*, among others.[20] Each communicates the same message. The inefficiency measure for $z$ is an upper bound for a $q-$ Nerlovian inefficiency. Let $\hat{q}$ solve (22), then

$$\left(\frac{q}{\hat{q}'g} - \frac{q}{q'g}\right)'z + \delta^*\left(\frac{q}{q'g}|Z\right) - \delta^*\left(\frac{\hat{q}}{\hat{q}'g}|Z\right)$$

closes any gap in (23) and gives a "real" (in units of the directional bundle $g$) of the length of that gap. For (22) picking the numeraire bundle, $g$, also determines the efficient direction and efficient projection. $D^Q(z|Z)$ is a scalar multiple of $g$ and $P^Q(z|Z)$ is the projection of $z$ onto $EffZ$ in the direction $g$ for all $z$.

More generally, however, $D^Q(z|Z)$ and $P^Q(z|Z)$ can vary with $z$. Another example illustrates. Let $Q = \left\{q \in \mathbb{R}^{S*} : \sum_{s\in 1,\ldots,S} |q_s| \leq 1\right\}$. Then

$$\begin{aligned}
\delta^Q(z|Z) &= \sup_q \left\{q'z - \delta^*(q|Z) : \sum_{s\in 1,\ldots,S} |q_s| \leq 1\right\} \\
&= \sup_q \left\{\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}'z - \delta^*\left(\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}|Z\right)\right\} \\
&= \inf_d \left\{\max_{s\in 1,\ldots,S} \{|d_s|\} : z - d \in Z\right\} \qquad (24)
\end{aligned}$$

The third equality follows by (15) since now $\delta^*(d|Q) = \max_{s\in 1,\ldots,S} \{|d_s|\}$. Thus, one can evocatively write

$$\delta^Q(z|Z) \geq \frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}'z - \delta^*\left(\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}|Z\right) \text{ for all } q, z. \tag{25}$$

[20]For $X \subset \mathbb{R}^S$ closed convex and containing the origin, its gauge and support functions are polar to one another. Hence,

$$\delta^*(q|X) = \sup\{q'x : \gamma(x|X) \leq 1\},$$

whence $\delta^*(q|X)\gamma(x|X) \geq 1$, which manifests Mähler's Inequality.

Without loss of generality, let the optimizer for (24) be $d_1$. One can now use subtraction in (25) to derive a dual inefficiency measure. But for the real units to comparable, the right-hand side of (25) needs to evaluated in units of $z_1$. Because the the element of $z$ that optimizes (24) can vary with $z$, the numeraire will vary with $z$.

Expressions (20), (23), and (25) all convey similar information. Expression (20) gives a dual inefficiency measure that closes the "gap" by simple subtraction, but it operates in what an economist calls "nominal" (albeit restricted to lie in $Q$) rather than "real" units. Expression (23) also gives a dual inefficiency measure by simple subtraction that is expressed in real terms, but it restricts $D^Q(z|Z)$. Expression (25) yields a real dual inefficiency measure by simple subtraction, but it requires identification of different numeraire for each application. Hence, dual inefficiency is not directly comparable across different $z$.

## Conjugacy Correspondences for Dual Inefficiency

Because it is closed convex in $q$ and closed convex in $z$, $\varphi^Q$ is what Rockefellar (1970, Section 29) refers to as a bi-function. That observation implies that $\varphi^Q$ has "a life of its own" as part of at least two well-defined conjugacy correspondences.

In particular, when viewed from a Lagrangean perspective

$$\varphi^Q(q, z|Z) = \delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q) - q'z$$

admits two parallel interpretations: one as the Lagrangean function for a closed convex program for minimizing $\delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q)$ over $q$; and the other as the Lagrangean function for a closed convex program for minimizing $\delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q)$ over $z$. In the first, $z$ plays the role of a Lagrange multiplier and $q$ plays that role in the second.

Define $\varphi^{Q**} : \mathbb{R}^S \times \mathbb{R}^S \to \bar{\mathbb{R}}$

$$\begin{aligned} \varphi^{Q**}(\bar{z}, z|Z) &\equiv \sup_{q \in \mathbb{R}^{S*}} \left\{ q'\bar{z} - \varphi^Q(q, z|Z) \right\} \\ &= \delta^Q(\bar{z} + z|Z) - \delta^Q(z|Z). \end{aligned} \tag{26}$$

Expression (26) defines the (partial) conjugate of $\varphi^Q(q, z|Z)$ treated as a closed convex function of $q$. The second equality follows by (11) and Proposition 2. We have:

**Proposition 5.** $\varphi^Q(q, z|Z) \overset{*}{\longleftrightarrow} \delta^Q(\bar{z} + z|Z) - \delta^Q(z|Z)$

$$\varphi^Q(q, z|Z) + \delta^Q(\bar{z} + z|Z) - \delta^Q(z|Z) \geq q'\bar{z} \;\; \forall q, \bar{z} \quad \textit{(Fenchel's Inequality)}$$

$$\hat{z} \in \partial_q \varphi^Q(q, z|Z) \Leftrightarrow q \in \partial_{\hat{z}} \delta^Q(\hat{z} + z|Z|) \Leftrightarrow \varphi^Q(q, z|Z) + \delta^Q(\bar{z} + z|Z) - \delta^Q(z|Z) = q'\hat{z}$$

Symmetrically, we define $\varphi^{Q*} : \mathbb{R}^{S*} \times \mathbb{R}^{S*} \to \bar{\mathbb{R}}$ as the (partial) conjugate of $\varphi^Q$ treated as a closed convex function of $z$:

$$\begin{aligned} \varphi^{Q*}(q, \bar{q}|Z) &\equiv \sup_{z \in \mathbb{R}^S} \left\{ \bar{q}'z - \varphi^Q(q, z|Z) \right\} \\ &= \delta^*(\bar{q} + q|Z) + \delta(\bar{q} + q|Q) - \delta^*(q|Z) - \delta(q|Q), \end{aligned} \tag{27}$$

where the second equality follows by (11) and Proposition 2. Hence,

**Proposition 6.** $\varphi^Q(q, z|Z) \overset{*}{\longleftrightarrow} \delta^*(\bar{q} + q|Z) + \delta(\bar{q} + q|Q) - \delta^*(q|Z) - \delta(q|Q)$

$$\varphi^Q(q, z|Z) + \delta^*(\bar{q} + q|Z) + \delta(\bar{q} + q|Q) - \delta^*(q|Z) - \delta(q|Q) \geq \bar{q}'z \;\; \forall \bar{q}, z \quad \textit{(Fenchel's Inequality)}$$

$$\begin{aligned} \hat{q} \in \partial_z \varphi^Q(q, zZ) \;\; &\Leftrightarrow \;\; z \in \partial_{\hat{q}} \delta^*(\hat{q} + q|Z) + \partial_{\hat{q}} \delta(\hat{q} + q|Q) \\ &\Updownarrow \\ \hat{q}'z - \delta^*(\hat{q} + q|Z) - \delta(\hat{q} + q|Q) \;\; &= \;\; \varphi^Q(q, z|Z) - \delta^*(q|Z) - \delta(q|Q) \end{aligned}$$

The dual-inefficiency measure is conjugate dual to a) a transformation of Paretian inefficiency and b) a transformation of Nerlovian inefficiency. Propositions 5 and 6 manifest the saddle-point theorem and are direct consequences of Lemma 1 and Proposition 2. Indeed, they convey the same mathematical information. Their force is that they establish that $\varphi^Q(q, z|Z)$, despite its frequent treatment as a residual, forms a component of a two conjugacy correspondences for two closed convex proper inefficiency measures. So, for example, specification of a $\varphi^Q(q, z|Z)$ that is closed convex in $q$ and closed convex in $z$ implies the existence of a well-behaved $\delta^Q(\bar{z} + q|Z) - \delta^Q(z|Z)$ and *vice versa* without the need for "...difficult constructive arguments" (McFadden 1978). Specification of a closed convex $\delta^Q(z|Z)$ implies the existence of a conjugate $\varphi^Q(q, z|Z)$ interpretable as a dual inefficiency measure. Parallel logic applies to $\delta^*(q|Z)$ and $\varphi^Q(q, z|Z)$. In short, if one can specify a meaningful $\delta^Q(z|Z)$, one can measure inefficiency and decompose it meaningfully without resorting to solving (11).

## Which Gap?

The introduction of Luenberger's (1992) measures into the inefficiency discussion emphasized that different directional orientations changed measures. Recognizing that choice matters, many contributions followed offering different perspectives and insights (Chambers et al. 1998; Tone 2001; Chambers and Färe 2004; Ray 2007; Cooper et al. 2011; Pastor et al. 2012; Aparicio et al. 2013, Zofio et al. 2013; Aparicio et al. 2015; Petersen 2018; Aparicio et al. 2023). Our discussion helps formalize a "folk theorem" that percolates through these contributions: *Dual inefficiency can be made as large or small as one chooses.* We demonstrate by first choosing $Q = \{\tilde{q}\}$ for $q \neq \tilde{q}$ to make $\varphi^Q(q, z|Z) = \infty$ and then choosing $Q = \{q\}$ to make $\varphi^Q(q, z|Z) = 0$.

How to use that information is unresolved. A marked difference between the "axiomatic approach to decision theory" and "axiomatic approach to inefficiency measurement" suggests one potential avenue. Both approaches use axioms and related mathematical machinery. But where the former emphasizes behavioral axioms imposed on preferences and induces functional forms, the latter uses axioms as "tests" that measures should pass. Our analysis shows that our Paretian inefficiency measures are a joint product of $Z$ and $Q$. It then follows that for a given $Z$, choosing $Q$ entails a choice of $\delta^Q$. We conjecture that a reasonable place to initiate a more structured approach to developing better measures is to investigate systematically the consequences of choosing "desirable" properties for $Q$.

In closing this discussion of dual inefficiency it seems wise to recall the cautionary advice of Charnes et al. (1978, p.443) in concluding their pathbreaking work.

> In private sector applications, the case for our proposed measure of efficiency begins to weaken to the extent that competition is present. In particular it begins to weaken as soon as freedom for the deployment of resources from one 'industry' to another (perhaps in a removed region) is present. Assessment of such possibilities would involve the introductioo of prices, or other weighting devices, for the evaluation of otherwise non-comparable alternatives.
>
> Although our measures are not designed for this sort of application they are designed for public sector programs in which the managers of various DMU's

are not free to divert resources to other programs merely because they are more profitable - or otherwise more attractive. Our measure is intended to evaluate the accomplishments, or resource conservation possibilities, for every DMU with the resources assigned to it. In golfing terminology it is, so to speak, a measure of 'distance' rather than 'direction' with respect to what has been (and might be) accomplished

At the risk of mixing golfing methaphors, our analysis emphasizes that measuring distance cannot be separated from direction. Or in perhaps in simpler terms, you cannot separate measured distance from the chosen yardstick.

# 3 Concluding Remarks

We pose inefficiency measurement as measuring the distance between an outcome and the efficient frontier for a closed convex set, $Z$. We define the efficient frontier using a generalized inequality, $\preceq_C$, that is a reflexive, transitive, and asymmetric convex binary relation, that partially orders $\mathbb{R}^S$, and that permits characterization of the efficient set using dual variates. We define Paretian inefficiency for an outcome's inefficiency as the maximal distance between the outcome and the support function for $Z$ while restricting dual variates to fall in a closed convex set. Thus, our formulation generalizes, while integrating into a common mathematic formulation, broad classes of existing inefficiency-measurement strategies. The result is a "minimal-support" inefficiency measure with characteristics that resemble the Nirnberg (1961) minimal-norm duality. We show that the resulting Paretian inefficiency measure is proper closed convex and conjugate dual to a restricted Nerlovian ineffiency measure that is proper closed convex. We use that conjugate duality to construct classes of dual composition rules for varying restrictions on $Z$ and dual-variate normalization.

We use the Paretian inefficiency measure to decompose measured Nerlovian inefficiency into a technical-inefficiency measure and a dual-inefficiency measure. The dual-inefficiency measure is a closed convex bi-function that satisifies two dual conjugacies: one with a differenced Paretian inefficiency measure and another with a differenced Nerlovian inefficiency measure. We use the dual-inefficiency measure to investigate recent concerns raised about the appropriateness of Nerlovian inefficiency decompositions.

In defining an inefficiency measure, we follow a trail blazed by Debreu (1951), Farrell (1957), Afriat (1972), and Charnes et al. (1978) that treats inefficiency measurement as evaluating an outcome in the most favorable dual terms. That setting encompasses broad classes of existing inefficiency measures. But other approaches exist (for example, Pastor (et al. 2007) and Färe et al. (2019)) and their mathematical formulation appears different from ours. Nevertheless these studies often yield measures similar to ours and other authors following our generic approach suggesting that formal identification of the ideal structural formulation for inefficiency measurement awaits further development.

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# Proofs

**Proposition** 1

a) Suppose that contrary to the claim that a $z^o \in \partial\delta^*(q|Z)$ for some $q \in ri\ C^*$ and a $z \in Z$ exist for which $z^o - z \in C\setminus\{0\}$. Then

$$\delta^*(z^o - z|C^*) = 0 > q'(z^o - z)$$

because $\delta^*(c|C^*) = \delta(c|C)$ for $c \in C$. But this violates the definition of $z^o$ as belonging to $\partial\delta^*(q|Z)$

b) By definition $z^o \in Eff Z \Rightarrow (z^o - Z) \cap C\setminus\{0\} = \emptyset$ so that $z^o - Z \subset \mathbb{R}^S$ and $C \subset \mathbb{R}^S$ are properly separated. Lemma 2 applies. Choose a $q \in ri\ C^*$ to obtain:

$$\begin{aligned}
\delta^*(q|z^o - Z) &= q'z^o - \delta^*(q|Z) \\
&\geq \inf\{q'x : x \in z^o - Z\} \\
&= 0 \\
&= \delta^*(q|C) \\
&> -\infty
\end{aligned}$$

as required.

**Lemma** 3

By Proposition 2 and (15):

$$q \in \delta^Q(z|Z) \Leftrightarrow \partial\delta(z - d|Z) \cap \partial\delta^*(d|Q) \Leftrightarrow \delta^Q(z|Z) = \delta(z - d|Z) + \delta^*(d|Q)$$

By (10), $dom\ \delta^*(\cdot|Q) = Q^*_\infty$ and the result follows because $\partial\delta^*(d|Q) = \emptyset$ for $d \notin dom\ \delta^*(\cdot|Q)$.

**Proposition** 3

a) Immediate from the properties of $\delta^*(q|Z)$ and (11).

b) By Proposition 2

$$z \in \partial\delta^*(q|Z) + d \text{ for } d \in \partial\delta(q|Q).$$

Lemma 3 implies $d \in C$. Thus, (10 ) and $\delta^Q(z|Z) \leq 0 \Rightarrow q'z \leq \delta^*(q|Z)$ for all $q$ establishing the result.

c) If $Q$ is bounded, then $Q_\infty = \{0\}$, whence $Q_\infty^* = \mathbb{R}^S$. If $\delta^Q(z|Z) > 0$ then $\delta^*(q|z - Z) > 0$ so that $z \notin Z$ by the separating hyperplane theorem.

**Proposition** 4

We start with a useful lemma. (See, for example, Hiriart-Urruty and LeMarëchal 2001 Proposition E.3.3.1.)

**Lemma 4.** *Let* $f(x) = \max_{j=1,\ldots,J}\{x'p^j - b_j\}$ *with* $p^j \in \mathbb{R}^{S*}$ *and* $b_j \in \mathbb{R}$ *for all* $j$*. Then for all* $p \in co\{p^1, \ldots, p^J\} = dom\ f^*$

$$f^*(p) = \min_\lambda \left\{ \sum_{j=1}^J \lambda_j b_j : \lambda \in \Delta_J \ \ p = \sum_{j=1}^J \lambda_j p^j \right\}$$

*where* $co\{\cdot\}$ *denotes the convex hull and* $\Delta_J \subset \mathbb{R}^J$ *denotes the unit simplex.*

We now prove the proposition

a) and b) are consequences of the conjugacy between addition and *infimal convolution.* (For example, Rockafellar 1970 Theorem 16.4, Hiriart-Urruty 2001 Theorem E.3.2.1).

c) Let $\delta^Q(z|Z) = \max_{j=1,\ldots,J}\{z'\bar{q}^j - b_j\}$. Then Lemma 4 requires for all $q \in co\{\bar{q}^1, \ldots, \bar{q}^J\} = dom\ \delta^{Q*}$ that

$$\delta^{Q*}(q|Z) = \min_\lambda \left\{ \sum_{j=1}^J \lambda_j b_j : \lambda \in \Delta_J \ \ q = \sum_{j=1}^J \lambda_j \bar{q}^j \right\}$$

d) Let $z^j \in \mathbb{R}^S,\ j = 1, \ldots, J$ and

$$\delta^*(q|Z) = \begin{cases} \max_{j=1,\ldots,J}\{q'z^j\} & \text{if } q \in C^* \\ \infty & \text{otherwise.} \end{cases}$$

Then using Lemma 4 gives

$$\delta(z^o|Z) = \begin{cases} 0 & \text{if } z^o - \sum_{j=1}^J \lambda_j z^j \in C, \lambda \in \Delta_J \\ \infty & \text{otherwise.} \end{cases}$$

Using (15)

$$\begin{aligned}
\delta^{Q}(z|Z) &= \inf_{z=z^o+d}\{\delta(z^o|Z)+\delta^*(d|Q)\} \\
&= \inf_{z=z^o+d}\left\{\begin{cases} 0 & \text{if } z^o \in \sum_{j=1}^{J}\lambda_j z^j + C, \lambda\in\Delta_J \\ \infty & \text{otherwise.}\end{cases} + \delta^*(d|Q)\right\} \\
&= \inf_{d,\lambda}\left\{\delta^*(d|Q) : z-d \in \sum_{j=1}^{J}\lambda_j z^j + C, \lambda \in \Delta_J\right\}
\end{aligned}$$

for $z \in dom\ \delta^Q$.

e) Let $Z = C$. Then

$$\delta(z|Z)\begin{cases} 0 & \text{if } z\in C \\ \infty & \text{otherwise.}\end{cases}$$

Applying (15) gives the result.

f) Let $Z = \bigcap\limits_k Z^k$ with each $Z^k \subset \mathbb{R}^S$, $k = 1, \ldots, K$ nonempty closed convex and $\bigcap\limits_k ri\ Z^k \neq \emptyset$. Then $\delta(z|Z) = 0 \Rightarrow \sum\limits_k \delta(z|Z^k) = 0$. Using the conjugacy between addition and infimal convolution gives

$$\delta^*(q|Z) = \inf_{q^1,\ldots,q^k}\left\{\sum_k \delta^*(q^k|Z^k) : \sum_k q^k = q\right\},$$

and the result then follows from (11).

g) Let $\delta^*(q|Z) = \sum\limits_n \delta^*(q|Z^k)$. Then the conjugacy between addition and infimal convolution gives

$$\delta(z|Z) = \inf_{z1,\ldots,z^K}\left\{\sum_k \delta(z^k|Z^k) : \sum_k z^k = z\right\},$$

and (15) gives

$$\delta^Q(z|Z) = \inf_{z^1,\ldots,z^K}\left\{\sum \delta(z^k|Z^k) + \delta^*\left(z - \sum_k z^k|Q\right)\right\}.$$

h) Let $Q = Q^1 \cap Q^2 \cap \cdots \cap Q^K$ with each $Q^k \subset \mathbb{R}^{S*}$ nonempty closed convex and $\bigcap\limits_k ri\ Q^k \neq \emptyset$. Then as in the proof of f)

$$\delta^*(d|Q^1\cap Q^2\cap\cdots\cap Q^K) = \inf\left\{\sum_k \delta^*(d^k|Q^k) : d = \sum_k d^k\right\},$$

and using (15) gives

$$\begin{aligned}\delta^Q(z|Z) &= \inf_{z=z^o+d}\left\{\delta(z^o|Z)+\inf\left\{\sum_k \delta^*\left(d^k|Q^k\right):d=\sum_k d^k\right\}\right\}\\ &= \inf_{d^1,\ldots,d^K}\left\{\delta\left(z-\sum_k d^k|Z\right)+\sum_k \delta^*\left(d^k|Q^k\right)\right\}\end{aligned}$$

.

i) Immediate from the assumption and (15).

j) Let $q^j \in \mathbb{R}^{S*}$, $j=1,\ldots,J$ and

$$\delta^*(d|Q)=\begin{cases}\max\limits_{j=1,\ldots,J}\{d'q^j\} & \text{if } d\in Q^*_\infty\\ \infty & \text{otherwise.}\end{cases}$$

Then as in the proof of d)

$$\delta(q|Q)=\begin{cases}0 & \text{if } q\in\sum_{j=1}^J \lambda_j q^j+Q_\infty, \lambda\in\Delta_J\\ \infty & \text{otherwise.}\end{cases}$$

and the result follows from (11).

k) Immediate.

l) Immediate.

m) Let $0\in Q$. Then, by the properties of gauge functions, $Q=\{q:\gamma(q|Q)\le 1\}$. The polarity between gauges and support functions then implies that

$$\begin{aligned}\delta^*(d|Q) &= \sup_q\{q'd:q\in Q\}\\ &= \gamma(d|Q_*)\end{aligned}$$

where $Q_*\equiv\left\{d\in\mathbb{R}^S:\delta^*(d|Q)\le 1\right\}$ is closed convex with $0\in Q_*$ (for example, Rockafellar 1970 Theorem 14.5). Using (15) gives the result.