Conjugate Paretian Inefficiency Measures

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Abstract

We study efficiency measurement using a partial ordering for the S-dimensional reals that generalizes the canonical less than or equal to partial ordering. We seek measures that judge outcomes as favorably as possible using a dual normalization strategy that generalizes those used in the minimum-norm and efficiency-measurement literatures.

The Paretian inefficiency measure minimizes the Nerlovian while constraining dual variates to fall in a predetermined closed convex set. We show that the Paretian inefficiency measure forms a dual conjugate pair with a restricted Nerlovian efficiency measure and that it generalizes the Nirnberg minimum-norm-minimal-distance duality. We use those results to develop conditions that ensure that the Paretian inefficiency measure is an exhaustive function (cardinal) representation of the feasible set. We then present a series of composition rules for different restrictions on the feasible set and dual-variate normalization that include generalizations of existing inefficiency measures.

We then treat Nerlovian inefficiency measurement and its decomposition in the presence of ex ante information information in the form of dual variates. Using Fenchel's Inequality, we decompose Nerlovian inefficiency into a technical-inefficiency component and a dual-inefficiency (allocative, price) component. We discuss the dual inefficiency measure's relevance and show that it satisfies two dual conjugacies.

Key words: Inefficiency measures, convex sets, conjugate duality, dual inefficiency, technical inefficiency

1 Introduction

Since their inceptions, operations research and economics have studied efficiency. Debreu (1951) early elaborated an analytical framework, and a series of subsequent studies developed tractable measurement algorithms (Farrell 1957, Nerlove 1965, Afriat 1972, Hanoch and Rothschild 1972, Charnes, Cooper, and Rhodes 1978, Färe and Lovell 1978, Banker, Charnes, and Cooper 1984), which in turn spawned a vast[1] empirical and conceptual literature. Charnes et al. (1978), however, marked an especial watershed. It showed that the Farrell-Afriat measurement schemes generalized to multiple-output settings solvable via linear programming, and its generalization by Banker et al. (1984) established *data-envelopment analysis* (DEA) as a canonical measurement framework.

Measuring efficiency requires a benchmark *efficient set* or *efficient frontier*. Here, efficiency measurement melds with the study of multiple-criteria (vector) optimization that seeks optimal solutions for *vector-valued* criterion functions. With intellectual roots tracing to Edgeworth (1881) and Pareto (1909), multiple-criteria problems often invoke the Paretian criterion and identify the efficient set with the *maximal set* for a subset of the partially ordered set $\left(\mathbb{R}^S, \leq\right)$, where $\leq$ is the canonical less than or equal to partial ordering (Ehrgott 2005, Löhne 2011). DEA studies often define their benchmark efficient set using the same criterion, as articulated formally by Banker et al.'s (1984) *Inefficiency Postulate* (for example Banker et al. 1984, Charnes, Cooper, Golany, Seiford, and Stutz 1985, Briec 1998, Briec and Lesourd 1999, Ray 2004 and 2007, Pastor, Lovell, and Aparicio 2012, Russell and Schworm 2009 and 2017, Mehdiloo and Podinovski 2021).

The Paretian criterion gives an efficient frontier that consists of a continuum of points. Because that leaves an infinity of potential reference points with which to compare a given outcome, the selection of an appropriate reference point remains unresolved. When outcomes are evaluated in existing markets, an obvious choice is the value-maximizing choice. Similarly, if the social optimum is easily identifiable, it provides the obvious reference point. But in many instances, such valuations are not available, and the analyst must choose a criterion for selecting a reference point.

[1]Far too voluminous to cite all contributions properly.

Debreu (1951, p. 273) suggested that the problem is to measure how far a non-optimal situation "...is from being optimal". Later Afriat (1972, p. 576) suggested representing "...operations as nearly efficient as possible". A large literature adopts this perspective and seeks measures that judge outcomes as favorably as possible. Here, efficiency measurement melds with the study of minimum-distance problems. By Nirnberg's Minimum Norm Duality Theorem (Nirenberg 1961, Luenberger 1969, Theorem 5.13.1), the minimal distance for a given norm between a point and a convex set's boundary is given by the maximal difference between the point and the set's support function with *dual variates restricted to the dual-norm's unit disc.* Many empirical efficiency measures seek to maximize differences between and an outcome and the feasible set's support function while restricting dual variates to, variously, a hyperplane, a closed half space, or to the intersection of closed half spaces (for example, Debreu 1951, Charnes et al. 1978, Luenberger 1992, Ray 2007, Pastor et al. 2012).

This paper studies efficiency measurement for a closed convex feasible set that generalizes the standard setting in several ways. We start by identifying the *Paretian (efficient) frontier* with the maximal set for a subset of the partially ordered set $\left(\mathbb{R}^S, \preceq_C\right)$ using a partial ordering, $\preceq_C$, that generalizes the canonical $\leq$ partial ordering. In a DEA setting, for example, that allows us to extend inefficiency-measurement beyond the traditional free disposal hull of an observed data set. In the absence of market-based outcomes, we follow Debreu (1951), Afriat (1972), and Charnes et al. (1978) and seek measures that judge outcomes as favorably as possible. But our normalization strategy generalizes those pursued in the minimum-norm and efficiency-measurement literatures by combining them under the more general restriction that dual variates belong to a nonempty closed convex subset of dual space that incorporates the unit disc, hyperplane, closed half spaces, and closed polyhedral sets as special cases. Phrasing the efficiency-measurement problem in these terms permits casting it in terms of the general theory of convex conjugates.

In what follows, we first define notation and recall some concepts from convex analysis. Then we specify the model and use vector-optimization results (Löhne 2011) to define an efficient frontier using $\preceq_C$ and to characterize it using dual methods (Proposition 1). Following Debreu (1951) and Nerlove (1965), we identify a dual *Nerlovian* inefficiency measure and use it to define a *Paretian* inefficiency measure as the solution to a convex programming problem. We show that the Paretian inefficiency measure forms a dual conjugate pair with a restricted Nerlovian efficiency measure (Proposition 2). Then we use the known conjugacy between the infimal-convolution and addition operations to show that the dual conjugacy generalizes the Nirnberg minimum-norm-minimal-distance duality. We use those results to develop conditions that ensure that the Paretian inefficiency measure is an exhaustive function (cardinal) representation of the feasible set (Proposition 3). That demonstration, when coupled with convexity of the inefficiency measure, ensures that the inefficiency measure provides the basis on which to measure differential movements along the efficient frontier associated with various elasticities and marginal values. We next present a series of composition rules for different restrictions on the feasible set and dual-variate normalization (Proposition 4). Special cases include generalizations of many inefficiency measures familiar from a DEA setting.

We then consider Nerlovian inefficiency measurement and its decomposition in the presence of *ex ante* information information in the form of dual variates on the social or market value of outcomes. Using Fenchel's Inequality, we examine the decomposition of Nerlovian inefficiency measured using *ex ante* information on dual variates into a *technical-inefficiency* component and a *dual-inefficiency* (allocative, price) component. We show that the dual-inefficiency measure is a closed convex bi-function in the sense of Rockafellar (1970, Section 29). We discuss the dual inefficiency measure, show its relevance for recent concerns raised about Nerlovian inefficiency decompositions, and show that it satisfies two dual conjugacies: a) one with a difference-based transformation of our Paretian inefficiency measure (Proposition 5); and b) one with a difference-based transformation of restricted Nerlovian inefficiency (Proposition 6).

Although we frame the analysis in more general terms than the polyhedral DEA setting, our results echo its familiar message that different choice criteria and different dual-normalization rules yield different efficiency measures. For example, Banker et al. (1984) derive input-oriented and output-oriented inefficiency measures by choosing different normalization criteria. Chambers, Chung, and Färe (1998) showed that the dual-variate normalization strategy of Luenberger (1992) yields inefficiency measures expressed in difference rather than ratio form. Thus, taxonomies often classify measures according to the functional structure (additive or multiplicative loss measure) of the criterion function or the orientation in which the outcome is compared to the benchmark set (slacks-based or path-based). Our results show that many of these different forms can be gathered under a common rubric that clarifies the essential mathematical issues yielding perceived differences. And because our results are established in a more general setting than DEA, it broadens the range of available measures while also showing that DEA-specific results established have implications for broader classes of measurement problems.

2 Notation and Preliminaries

Let $\bar{\mathbb{R}} = [-\infty, \infty]$. The *effective domain* for a function $f : \mathbb{R}^S \to \bar{\mathbb{R}}$, *dom* f, is

$$dom\ f \equiv \left\{x \in \mathbb{R}^S : f(x) < \infty\right\},$$

its *subdifferential correspondence*, $\partial f : \mathbb{R}^S \rightrightarrows \mathbb{R}^{S*}$, is[2]

$$\partial f(x) \equiv \left\{q \in \mathbb{R}^{S*} : q'(z-x) \leq f(z) - f(x),\ \forall z \in \mathbb{R}^S\right\},$$

and its (one-sided) *directional derivative* in the direction $n \in \mathbb{R}^S$, $D^n \cdot f : R^S \times R^S \to \bar{\mathbb{R}}$, is

$$D^n \cdot f(x) \equiv \lim_{\lambda \downarrow 0} \frac{f(x+\lambda n) - f(x)}{\lambda}.$$

For f proper convex[3], $\partial f(x) \neq \emptyset$ for $x \in ri\,(dom f)$, $D^n \cdot f$ is proper closed[4] convex as a function of n and

$$D^n \cdot f(x) = \sup\{q'n : q \in \partial f(x)\}. \tag{1}$$

The (convex) *conjugate* for f, $f^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, is[5]

$$f^*(q) \equiv \sup_{x \in \mathbb{R}^S} \{q'x - f(x)\}. \tag{2}$$

[2]$\mathbb{R}^S$ is, of course, self-dual. We retain the notation, $\mathbb{R}^{S*}$, for its dual space to ensure a clear distinction between dual and primal variates.

[3]A convex function f is *proper* if *dom* f is nonempty, and $f(x) > -\infty$ for all x.

[4]A function is closed if its closure is the function itself. For proper convex functions, closedness is equivalent to lower semi-continuity (Rockafellar 1970, p. 52).

[5]Moreau (1966) calls f^* *la fonction polaire* to f. Some writers call it the *Fenchel transform.* Rockafellar and Wets (2009) call it the *Legendre-Fenchel transform.*

f^* is closed convex. For f proper closed convex:

$$\begin{aligned} f^{**}(x) &\equiv \sup_{q\in\mathbb{R}^{S*}} \{q'x - f^*(q)\} \\ &= f(x), \end{aligned} \tag{3}$$

and f^* is also proper. Expressions (2) and (3) form the *conjugacy correspondence*, $f \overset{*}{\longleftrightarrow} f^*$, between proper closed convex f and its proper closed convex conjugate f^*. A well-known consequence is (see, for example, Rockafellar 1970, Moreau 1966, Rockafellar and Wets 2009, Aubin and Ekeland 2007, Bertsekas 2009)

Lemma 1. *Let f be proper closed convex. Then f^* is proper closed convex,*

$$f(x) + f^*(q) \geq q'x \ \ \forall q, x \quad \textit{(Fenchel's Inequality)} \tag{4}$$

$$\hat{q} \in \partial f(\hat{x}) \Leftrightarrow \hat{x} \in \partial f^*(\hat{q}) \Leftrightarrow f(\hat{x}) + f^*(\hat{q}) = \hat{q}'\hat{x}, \tag{5}$$

$$\partial f^*(q) = argmax_x \{q'x - f(x)\} \quad \partial f(x) = argmax_q \{q'x - f^*(q)\}. \tag{6}$$

Let $ri\ X$ denote the relative interior of $X \subset \mathbb{R}^S$. Define the *indicator function*, $\delta : \mathbb{R}^S \to \{0, \infty\}$, for $X \subset \mathbb{R}^S$ by

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ \infty & \text{otherwise.} \end{cases} \tag{7}$$

For $X \subset \mathbb{R}^S$ closed convex and nonempty, $\delta(x|X)$ is proper closed and convex. The *support function*, $\delta^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, for X is

$$\begin{aligned} \delta^*(q|X) &\equiv \sup\{q'x : x \in X\} \\ &= \sup_{x\in\mathbb{R}^S} \{q'x - \delta(x|X)\}. \end{aligned} \tag{8}$$

δ^*, as the conjugate of δ, is closed and sublinear. If X is closed nonempty and convex, expression (3) implies δ^* is proper and

$$\delta(x|X) = \sup_{q\in\mathbb{R}^{S*}} \{q'x - \delta^*(q|X)\}. \tag{9}$$

The *gauge function* for $X \subset \mathbb{R}^S$, $\gamma : \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined

$$\gamma(x|X) = \inf\{\gamma > 0 : x \in \gamma X\}. \tag{10}$$

If X is nonempty closed convex and $0 \in X$,

$$X = \{x : \gamma(x|X) \leq 1\}.$$

The *negative polar cone*[6] of a convex set $X \subset \mathbb{R}^S$ is

$$X^* \equiv \left\{q \in \mathbb{R}^{S*} : q'x \leq 0, \forall x \in X\right\}.$$

For X a closed convex nonempty cone $X^{**} = X$. The *polar set* of $X \subset \mathbb{R}^S$ is

$$X^o \equiv \left\{q \in \mathbb{R}^{S*} : q'x \leq 1, \forall x \in X\right\}.$$

The *recession (asymptotic) cone* of $X \subset \mathbb{R}^S$ is

$$X_\infty \equiv \left\{d \in \mathbb{R}^S : X + \beta d \subset X, \beta \geq 0\right\}.$$

$0 \in X_\infty$ for all $X \subset \mathbb{R}^S$. For X closed convex and nonempty :

$$\begin{aligned} \textit{cl dom } \delta^*(\cdot|X) &= X^*_\infty \\ (\textit{cl dom } \delta^*(\cdot|X))^* &= X_\infty \end{aligned} \tag{11}$$

(Hiriart-Urruty and LeMaréchal 2001, Proposition C.2.2.4). And for X closed convex with $0 \in X$:

$$X_\infty = \left\{x \in \mathbb{R}^S : \gamma(x|X) = 0\right\}$$

(Hiriart-Urruty and LeMaréchal 2001, Theorem C.1.2.5).

A useful separation result is:

Lemma 2. *(Rockafellar 1970) Let X^1 and X^2 be nonempty convex subsets of $\mathbb{R}^S$ that satisfy $ri\ X^1 \cap ri\ X^2 = \emptyset$, then there exists $q \in \mathbb{R}^{S*}$ such that*

$$\begin{aligned} \inf\left\{q'x : x \in X^1\right\} &\geq \delta^*\left(q|X^2\right) \\ \delta^*\left(q|X^1\right) &> \inf\left\{q'x : x \in X^2\right\}. \end{aligned}$$

[6]Some authors refer to X^* as the polar cone of X.

Efficient Outcomes

The Feasible Set

Feasible outcomes are given by a closed nonempty convex set $Z \subset \mathbb{R}^S$. Z admits different interpretations including, among others: an input set, an output set, a technology set, a state-contingent technology set, and a (convex) envelope of observed data points. For purposes of a concrete discussion we refer to Z as the *technology.* Elements of Z are *netputs.*

Working with netputs differs from studies that segregate inputs from outputs. The notational difference promotes simplicity and generality and accommodates the potential for intermediate outputs in the technology. It avoids the practical difficulties encountered in applications in segregating inputs from outputs when one decisionmaker is a net producer of, say, "corn and hogs" and another facing the same Z is a net user of "corn" and produces only " hogs", or in instance such as those illustrated by Mehdiloo and Podinovski (2024) where the demarcation is not obvious. Reconfiguration into inputs and outputs is always possible by setting $z = (-x, y)$ where $x \in \mathbb{R}^N_+$ denote inputs and $y \in \mathbb{R}^M_+$ denote outputs and $S = M + N$.

The Efficient Set

Inefficiency-measurement and vector-optimization studies often use the canonical $\leq$ partial ordering of $\mathbb{R}^S$ (for example, Banker et al. 1984, Charnes et al. 1985, Ehrgott 2005, Ray 2004, Pastor et al. 2012, Russell and Schworm 2009 and 2017) and the Paretian criterion to identify the efficient frontier (see, for example, Banker et al.'s (1984, p. 1081) *Inefficiency Postulate*). That choice limits applicability of the resulting measures and can conflict with physical reality in applied settings. For example, it rules out bounded Z including the Charnes et al. (1985) *empirical production set* (their expression 3.1), well-documented instances of input or output congestion, the presence of by-products, and can create material-balance specifications that contradict the first law of thermodynamics among other concerns (see, for example, Shephard 1970, Färe, Grosskopf, and Lovell 1985, Byrnes, Färe, Grosskopf, and Lovell 1988, Kuosmanen 2005, Mehdiloo and Podinovski 2019 and 2024, Kao and Hwang

2021).

We use a more general binary relation, $\preceq_C$, defined by

$$x \preceq_C y \Leftrightarrow x - y \in C,$$

where $C \subset \mathbb{R}^S$ is a closed pointed convex cone. $\preceq_C$ is reflexive, transitive, and antisymmetric so that $\left(\mathbb{R}^S, \preceq_C\right)$ forms a *partially ordered set* (Boyd and Vandenberghe 2004, Nemirovski 2007, Löhne 2011).

To relate this partial ordering to Z, we posit an axiom:

Axiom 1. *$Z_\infty = C$ (where $C \subset \mathbb{R}^S$ defines $\preceq_C$).*

When Z is interpreted as a technology set, its recession cone, Z_∞, describes the directions in which starting at a feasible netput in Z, one can move towards infinity while maintaining feasibility. Thus, it accords with the production-theoretic notion of netput-disposability.

Example 1. *Let $C = \mathbb{R}^S_-$. Then Z satisfies the canonical Banker et al. (1984) Inefficiency Postulate, which requires that $z \in Z \Rightarrow z' \in Z$ for $z' \leq z$. Inputs can be always be feasibly increased and outputs feasibly decreased.*

Example 2. *Let $C = \{0\}$. Then Z is compact. Special cases include the (compact) weak-disposable, convex hull technologies and the empirical production set of Charnes et al. (1985).*

Example 3. *Let $S = N + M$. Then partition z as $\left(z^N, z^M\right)'$ with $z^N \in \mathbb{R}^N$ and $z^M \in \mathbb{R}^M$ and let $C = \left(\mathbb{R}^N_-, 0\right)'$ to obtain a generalized hybrid technology that exhibits selective strong disposability (Mehdiloo and Podinovski 2019 and 2021).*

Example 4. *Let $C = \{d\}$ where $d \in \mathbb{R}^S$. Then Z satisfies the "goodness in the numeraire (d)" in the direction d criterion (Chambers and Färe 2022).*

Using Axiom 1 and the Paretian criterion, we define the efficient set as:[7]

Definition 1. *The efficient subset, $EffZ$, of $(Z, \preceq_C)$ is*

$$EffZ \equiv \{z^o \in Z : \nexists z \in Z \text{ for which } z^o \preceq_C z \wedge z \neq z^o\}.$$

[7] Alternatively, z^o is efficient if and only if no $z \in Z$ exists for which $z^o - z \in C \backslash \{0\}$.

Because Z is closed convex, $\delta^*(q|Z)$ is closed sublinear. Thus, Axiom 1 and (11) imply that $dom\ \delta^*(\cdot|Z) = C^*$. We use that observation to state a result that extends those for dual representations of efficient sets for the canonical $\leq$ partial ordering (for example, Charnes et al. 1985, Ehrgott 2005, Theorem 3.6 and Corollary 3.7) to $\preceq_C$:[8]

Proposition 1. *a)* $\partial\delta^*(q|Z) \subset EffZ$ *for all* $q \in ri\ C^*$. *b)* $z^o \in EffZ \Rightarrow z^o \in \partial\delta^*(q|Z)$ *for some* $q \in ri\ C^*$.

Proof: See Appendix.

Remark 1. *Proposition 1.a remains true for general closed* Z*, but part b) requires convexity. A geometric interpretation of Proposition 1 is that* $EffZ$ *corresponds to the set of faces of* Z *exposed by* $q \in ri\ C^*$.

In an economic setting where competitive firms maximize profit, the connection between $\partial\delta^*(q|Z)$, as the profit-maximizing netput vectors, and $Eff\ \ Z$ is familiar. In a broader context, the connection between $\partial\delta^*(q|Z)$ and $Eff\ \ Z$ helps explain the primacy of linear scalarization techniques in solving vector-optimization problems (Ehrgott 2005, Löhne 2011).

Inefficiency Measures

Measures Defined

We call

$$q'z - \delta^*(q|Z) = \delta^*(q|z - Z)$$

the *q-Nerlove efficiency measure for z.* When the dual variates, $q \in \mathbb{R}^{S*}$, are prices or shadow prices, $\delta^*(q|z - Z)$ measures excess cost, foregone revenue, or foregone profit and is (minus) Nerlove's (1965) efficiency measure.[9] Because $\mathbb{R}^{S*}$ is the space of linear functionals on $\mathbb{R}^S$,

[8] Proposition 1 can be inferred, for example, from Theorems 4.1 and 4.2 in Löhne (2011). We present a direct proof, which follows standard arguments, in an Appendix to ensure a self-contained treatment. Note the obvious connection to the First and Second Welfare Theorems of Economics.

[9] Debreu (1951) uses the term 'dead loss'. Some writers, following the tradition established in economics, treat the negative of our measure. We chose our approach to emphasize the inherent connection between inefficiency measurement and the theory of convex conjugates.

$\delta^*(q|z-Z)$ measures the distance between the hyperplanes with normals q that, respectively, include z and that support Z. Regardless of interpretation, $\delta^*(\cdot|z-Z) \overset{*}{\longleftrightarrow} \delta(\cdot|z-Z)$. The Nerlove efficiency measure, as the support function for a translated convex set $z-Z$, is dual to $\delta(z|z-Z)$.

By definition, $\delta^*(q|z-Z) \leq 0$ for all $z \in Z$. We say that $z \in Z$ is *q-Nerlove inefficient* when $\delta^*(q|z-Z) < 0$. Dual Nerlovian measures, therefore, can distinguish between *inefficient* points lying inside Z and its *efficient* boundary points. Because $(Z, \preceq_C)$ is a partially ordered set, one can encounter situations where a given z is $\hat{q}$-Nerlove inefficient but not q^o-Nerlove inefficient for $\hat{q} \neq q^o$. To accommodate such outcomes, we have

Definition 2. *$z \in Z$ is Pareto inefficient if and only if $\delta^*(q|z-Z) < 0$ for all $q \in ri\ C^*$.*

If $z \in Z$ is Pareto inefficient, then

$$q'z < \delta^*(q|Z)$$

for all q with $\delta^*(q|Z)+\delta^*(-q|Z) > 0$. Thus, Paretian inefficiency requires that $z \in ri\ Z$ (for example, Hiriart-Urruty and LeMaréchal 2001, Theorem C.2.2.3). A well-known stumbling block to designing an algorithm to *measure* Paretian inefficiency is that dual variates are determined only up to multiplication by a positive scalar. Traditional resolutions include restricting q to the level set for a linear function of q or to its associated closed half space.

Debreu (1951, p.284), for example, suggests "... dividing by a price index" and thus chooses the dual value of z as a numeraire. Charnes et al. (1978) require that a subvector of z, which they term inputs, have a dual value of 1. Luenberger (1992) requires that the dual value of a predetermined element of $\mathbb{R}^S$, *the direction* of Chambers, Chung, and Färe (1996,1998), be at least 1. Ray (2007) extended the Luenberger approach by requiring that the dual value of two subvectors of z corresponding to the observed inputs and outputs, respectively, at least equal 1. That, in turn, implies that virtual profit for the observed inputs and outputs equals zero. Pastor et al. (2012) generalize Ray (2007) by treating the case of an arbitrary number of linear inequalities. Petersen (2018, p. 1071) considers setting the direction "...equal to price vector...normalized to a unit vector".

The Minimum Norm Duality Theorem (Nirenberg 1961, Luenberger 1969, Theorem 5.13.1) manifests a different normalization. It shows that the minimal distance between a point and the boundary of a convex set for a given norm is the maximal difference between the hyperplane through that point and the set's support function with dual variates restricted to lie within the dual norm's unit disc

We incorporate the inefficiency-measurement literature approaches and the minimum-norm approach under a more general rubric and require that q belong to a closed convex nonempty $Q \subset \mathbb{R}^{S*}$. Our *measure of Pareto Inefficiency*, $\delta^{Q} : \mathbb{R}^{S} \to \bar{\mathbb{R}}$, is defined relative to Q as:

$$
\begin{aligned}
\delta^{Q}(z|Z) &\equiv \sup\{q'z - \delta^{*}(q|Z) : q \in Q\} \\
&= \sup_{q \in \mathbb{R}^{S*}} \{q'z - \delta^{*}(q|Z) - \delta(q|Q)\}.
\end{aligned}
\tag{12}
$$

$\delta^{Q}(z|Z)$ isolates the element(s) of Q for which the normalized q-Nerlove inefficiency is "...as nearly efficient as possible" (Afriat 1972). The superscript notation reminds us that δ^{Q} also has an interpretation as a "restricted" indicator function.

We choose (12) as the criterion function because of its longstanding importance in inefficiency measurement, its connection to minimum-norm problems, and its mathematical links to the conjugacy correspondence $\delta^{*} \overset{*}{\longleftrightarrow} \delta$ (for example, Debreu 1951, Nirenberg 1961, Luenberger 1969, Afriat 1972, Charnes et al. 1978, Ray 2004 and 2007, Pastor et al. 2012, among others). Nevertheless, other writers use different criteria to induce or to rationalize inefficiency measures that include some of the measures induced below. For example, some impose stronger domain restrictions than ours and seek measures that maximize, for example, the ratio of revenue to cost, the ratio of realized revenue to maximal revenue, and the ratio of minimal cost to realized cost. (Note that it is routine in such settings to follow Charnes et al. (1978) and Tone (2001) and convert the resulting fractional programs into linear programs by setting the denominator of the fractional objective function to one.) Pastor, Ruiz, and Sirvent (2007) use a prespecified measure of closeness. Färe, He, Li, and Zelenyuk (2019), Färe and Zelenyuk (2020), and Zelenyuk and Zhao (2024) study an approach that unifies different measures under a common framework that involves maximizing a generic function of weights, θ, for observed dual variates, z^{0}, and primal variates, q^{0}, subject to a constraint that the reference point belongs to the intersection of our Z and a constraint set

C

$$\sup_{z,\theta}\Big\{ f(\theta) : (\theta \cdot q^0)' z^0 \leq (\theta \cdot q^0)' z, z \in Z \cap C \Big\},$$

where $q \cdot x$ denotes the Hadamard product of q and x. By varying, f and C, one can obtain Farrell-oriented profit-based inefficiency measures, versions of Russell inefficiency measures, and a variety of slack-based inefficiency measures.

A Conjugacy Result

Lemma 1 and (12) give:

Proposition 2. $\delta^Q : \mathbb{R}^S \to \bar{\mathbb{R}}$ *is proper closed convex. Moreover,*

$$\delta^Q (z|Z) \overset{*}{\longleftrightarrow} \delta^* (q|Z) + \delta (q|Q) \tag{13}$$

$$\delta^Q (z|Z) + \delta^* (q|Z) + \delta (q|Q) \geq q' z \;\; \forall q, z \tag{14}$$

$$\hat{q} \in \partial \delta^Q (\hat{z}|Z) \Leftrightarrow \hat{z} \in \partial \delta^* (\hat{q}|Z) + \partial \delta (\hat{q}|Q) \Leftrightarrow \delta^Q (\hat{z}|Z) + \delta^* (\hat{q}|Z) + \delta (\hat{q}|Q) = \hat{q}' \hat{z}, \tag{15}$$

By construction, $\delta^Q (z|Z)$ is the conjugate of $\delta^* (q|Z) + \delta (q|Q)$ ensuring that it is a closed convex function of z. On the other hand, $\delta^* (q|Z) + \delta(q|Q)$ is proper closed convex which with Lemma 1 establishes that δ^Q is proper and that $\delta^* (q|Z) + \delta (q|Q) \overset{*}{\longleftrightarrow} \delta^Q (z|Z)$. The remainder of the proposition also follows from Lemma 1.

Proposition 2 establishes a conjugacy correspondence between a mapping, δ^Q, defined on $\mathbb{R}^S$ and one defined on its dual space $\mathbb{R}^{S*}$ that provides a description for how the choice of Q affects δ^Q. Following Debreu (1951), Charnes et al. (1978), Ray (2007), and others we have cast inefficiency measurement as a problem of choosing dual variates, $q \in Q$. But just as primal and dual algorithms exist for linear programs, we can reformulate (12) as optimizing over $\mathbb{R}^S$. The well-known conjugacy between the operations of *addition* and *infimal convolution* gives

$$\begin{aligned} \delta^Q (z|Z) &= (\delta^* (q|Z) + \delta (q|Q))^* (z) \\ &= \underset{z = z^o + d}{inf} \{\delta (z^o|Z) + \delta^* (d|Q)\} \\ &= \inf_{d \in \mathbb{R}^S} \{\delta^* (d|Q) : z - d \in Z\} \end{aligned} \tag{16}$$

where $\underset{z=z^o+d}{inf} \{\delta(z^o|Z) + \delta^*(d|Q)\}$ defines the *infimal convolution* of δ and δ^* (for example, Rockafellar 1970 Theorem 16.4, Hiriart-Urruty 2001 Theorem E.3.2.1).(Some authors call the infimal convolution operation *epi-addition*. See, for example, Rockafellar and Wets 2009.)

Constructing the inefficiency measure, thus, reduces to locating the "minimal translation" of z that ensures that it belongs to Z, where the degree of minimality is measured by the support function for Q. The conjugate correspondence established, therefore, manifests a "minimal-support duality" that involves both Z and Q. Geometrically, we translate z until a tangency occurs between the boundaries of Z and a level set for $\delta^*(d|Q)$. We reformulate (15) in equivalent terms as

$$(17) \quad q \in \partial\delta(z-d|Z) \cap \partial\delta^*(d|Q) \Leftrightarrow q \in \delta^Q(z|Z) \Leftrightarrow \delta^Q(z|Z) = \delta(z-d|Z) + \delta^*(d|Q).$$

The conjugate manifestations of δ^Q as a (normalized) minimal difference between $q'z$ and $\delta^*(q|Z)$ and the infimal convolution of $\delta(\cdot|Z)$ and $\delta^*(\cdot|Q)$ mirror the optimization principles behind Luenberger's (1992) demonstration that benefit and shortage functions support calculation of Paretian efficient outcomes for a competitive market. Where Luenberger's demonstration reaffirms that Paretian-efficient calculations reflect assumptions on preferences and the technology, Proposition 2 shows that "Paretian inefficiency measurement" reflects assumptions on Z and the numeraire embedded in the choice of Q. δ^Q is, in essence, a "joint product" of Z and Q. The far left-hand side of (17),

$$q \in \partial\delta(z-d|Z) \cap \partial\delta^*(d|Q),$$

which requires that the subdifferentials of $\delta(z^o|Z)$ and $\delta^*(z-z^o|Q)$ overlap (tangencies between boundaries), manifests the *Dubovitskii-Milyutin Lemma* that characterizes extremals set-valued optimization problems (Isac and Khan 2008).

Cardinal Representations of Z and Marginal Values

Färe and Lovell (1978) showed the formal equivalence between Farrell's (1957) inefficiency score and Shephard's (1953, 1970) distance function. Because Shephard's distance functions characterize weakly disposable technology sets, Färe and Lovell's result establishes that inefficiency measures can play two roles: one as measuring performance efficiency and another as mathematically characterizing technology sets. In fact, Shephard's distance function can trace its mathematical lineage to Minkowski's (1911) *Distanzfunktion* (Newman 1987). And Minkowski-type functionals (gauges, co-gauges (distance), s-gauges) often appear in mathematics as cardinal (function) representations of star-shaped and convex sets (for example, Moreau 1966, Rockafellar 1970, Aliprantis and Border 2007). [10]

Such cardinal representations also formalize the economics notion of a "transformation function" used to identify production possibilities sets and efficient production frontiers. Transformation functions are key to framing economic models of individual decision-makers responding to technological constraints as well as framing differing notions of elasticities associated with the slope of an efficient frontiers (Podinovski and Førsund 2010) and measures of economies of scale.

Not all inefficiency measures, however, provide cardinal representations of their associated Z. We show that a key issue is to ensure that the choice Q is consonant with $C(Z^{\infty})$. Following Ray (2007) while using (17), define:

$$D^{Q}(z|Z) \equiv \left\{d : \delta^{*}(d|Q) \cap \partial\delta^{Q}(z|Z) \neq \emptyset\right\} \tag{18}$$

as the *endogenous directions* for (12) and

$$P^{Q}(z|Z) \equiv \left\{z - d : d \in D^{Q}(z|Z)\right\}, \tag{19}$$

as its *endogenous projections*.

We have:

Lemma 3. *Let (12) have a finite solution. Then $D^{Q}(z|Z) \subset Q^{*}_{\infty}$ for all z.*

Proof: See Appendix.

Proposition 3. *Let (12) have a finite solution.*

a) $z \in Z \Rightarrow \delta^{Q}(z|Z) \leq 0$;

b) if $Q^{*}_{\infty} \subset C$ then $\delta^{Q}(z|Z) \leq 0 \Rightarrow z \in Z$; and

c) if Q is bounded, $\delta^{*}(q|z - Z) > 0 \Rightarrow z \notin Z$.

[10] These functionals are all based on some measure of a point's distance from a set's boundary, their interpretation as inefficiency measures is natural. Indeed, Newman (1987) introduces his survey of gauge functions in economics by calling them "...sensible measure(s) of efficiency".

Proof: See Appendix.

By Proposition 3, choosing $Q^*_\infty \subset C$ ensures that $z \in Z \Leftrightarrow \delta^Q(z|Z) \leq 0$. Thus, movements along the efficient frontier can be identified with differential movements along the 0-level set of $\delta^Q(z|Z)$.

Partition z as $z = (z^0, z^1)'$. Proposition 2 ensures that δ^Q is convex so that $D^{\left(\alpha\bar{z}^0,\beta\bar{z}^1\right)} \cdot \delta^Q(z|Z)$ exists everywhere in $ri\ dom\left(\delta^Q\right)$ with

$$D^{\left(\alpha\bar{z}^0,\beta\bar{z}^1\right)} \cdot \delta^Q\left(z^0, z^1|Z\right) = \sup\left\{\alpha q^{0\prime}\bar{z}^0 + \beta q^{1\prime}\bar{z}^1 : \left(q^0, q^1\right)' \in \partial\delta^Q(z|Z)\right\}, \tag{20}$$

for the direction $n = (\alpha\bar{z}^0, \beta\bar{z}^1)$ with $\alpha, \beta \in \mathbb{R}_{++}$. By the sublinearity of (20) in n

$$D^{\left(\alpha\bar{z}^0,\beta\bar{z}^1\right)} \cdot \delta^Q\left(z^0, z^1|Z\right) = \alpha D^{\left(\bar{z}^0,\frac{\beta}{\alpha}\bar{z}^1\right)} \cdot \delta^Q\left(z^0, z^1|Z\right) = \beta D^{\left(\frac{\alpha}{\beta}\bar{z}^0,\bar{z}^1\right)} \cdot \delta^Q\left(z^0, z^1|Z\right). \tag{21}$$

Because $\delta^Q(z|Z)$ is closed convex, it is almost everywhere differentiable on $ri\ dom\ \delta^Q$. That allows us to rewrite (20) (almost everywhere) as

$$D^{\left(\alpha\bar{z}^0,\beta\bar{z}^1\right)} \cdot \delta^Q\left(z^0, z^1|Z\right) = \alpha\left(\frac{\partial\delta^Q}{\partial z^0}\right)'\bar{z}^0 + \beta\left(\frac{\partial\delta^Q}{\partial\bar{z}^1}\right)'z^1, \tag{22}$$

where $\frac{\partial\delta^Q}{\partial z^k}$ denotes the gradient of δ^Q with respect to sub-vector z^k. Setting (22) equal to zero and evaluating at $z \in Eff\ Z$, gives

$$\frac{\alpha}{\beta} = -\frac{\left(\frac{\partial\delta^Q}{\partial\bar{z}^1}\right)'z^1}{\left(\frac{\partial\delta^Q}{\partial z^0}\right)'\bar{z}^0} \tag{23}$$

as the proportional marginal change in z^0 in the direction $\bar{z}^0$ that balances a marginal change in z^1 in the direction $\bar{z}^1$ along the efficient frontier. The familiar notions of marginal products, input elasticities, output elasticities, and the scale elasticity are all special cases of (23). For the points of measure zero on which δ^Q is not smooth, one can follow Chambers and Färe's (2008) adaptation of Rockafellar's (1970) (one-sided) directional derivatives to non-smooth structures to obtain (one-sided) marginal values and elasticities at $(z^0, z^1)' \in Eff\ Z$ by setting the appropriate version of (20) to zero and solving for appropriate $\frac{\alpha}{\beta}$. (Podinovski and Førsund (2010) and Podinovski et al. (2016) provide linear programs for calculating the scale elasticity and one other one-sided measures in the DEA setting.)

Antecedents

Propositions 2 and 3 have antecedents in both the broader optimization literature and in the narrower inefficiency-measurement literature. Nirenberg's *Minimum-Norm Duality Theorem* (Nirenberg 1961 and Luenberger 1969) is of particular note. As we discuss below, the support and the gauge functions for a closed convex set are polar to one other. And gauge functions for compact zero-symmetrical sets on $\mathbb{R}^S$, in turn, form a one-to-one correspondence with their norms (see, for example, Rockafellar 1970 Section 15). Hence, (16) encompasses minimum-norm measures as special cases. Briec (1998) and Briec and Lesourd (1999) generalize the Hölder norm to define a *Hölder distance function* that seeks to measure "...the smallest required modifications in..." netputs to attain the efficient frontier. Working in partially-ordered infinite-dimensional space, Bator and Briec (2024) study the interrelationships between directional distance functions, normed distance functions, and Nirnberg's Theorem.

Different studies show that changing the price normalization changes the resulting measure. For example, early authors recognized that normalizing the dual value of "inputs" and normalizing the dual value of "outputs" gave different inefficiency measures. Following Luenberger (1992), Chambers, Chung, and Färe (1996, 1998) generalized that observation to distinguish between radial input and output measures, input-directional distance functions, output-directional distance functions, and technology-directional distance functions. Ray (2007) generalized further by normalizing input and output vectors separately (also see Aparicio, Pastor, and Ray 2013). Cooper, Pastor, Aparicio, and Borras (2011) relate the Russell efficiency measure to (minus) the l^∞ norm. Pastor et al. (2012) show that minimizing $\delta^* (q|z - Z)$ subject to different normalizations yields different inefficiency measures. And by restricting attention to the canonical DEA model and normalization conditions "...represented by means of a finite set of equalities and/or inequalities...", they induce versions of the Banker et al. (1984) measure, a directional distance function, the weighted-additive measure, and the Russell measure.[11] Färe, Grosskopf, and Whitaker (2013) derive an *endoge-*

[11]Because Pastor et al. (2012) do not impose convexity on their normalizing set in their general model, it is not a special case of ours. But the listed representation results are all developed for the polyhedral case which is covered by Proposition 2.

nous directional measure. Aparicio, Borras, Pastor, and Vidal (2015) show that the Russell efficiency measure is conjugate to the revenue function subject to dual variates falling in Q that is a special case of Proposition 4.h below. Chambers (2024) develops a related inefficiency measure that requires the dual variates to fall on the level set for a sublinear function of q and presents a partial version of Proposition 2.

Conjugate Inefficiency Correspondences

Our analysis starts with Z, isolates $EffZ$, and then uses (12) to measure inefficiency. Proposition 2 implies that an equivalent, equally relevant, dual approach exists. One can start with a proper closed convex inefficiency measure, $\delta^Q(z|Z)$, and then use the conjugacy correspondence to resurrect a conjugate $\delta^*(q|Z) + \delta(q|Q)$ that is proper closed and convex. Proposition 2 ensures the induced $\delta^*(q|Z) + \delta(q|Q)' s$ consistency with $\delta^Q(z|Z)$ without the need for "...difficult constructive arguments" (McFadden 1978). Broad classes of functions are closed convex. By Proposition 2, each such class defines a class of inefficiency measures and a conjugate dual class of $\delta^*(q|Z) + \delta(q|Q)' s$.

Example 5. *Let* $\delta^Q(z|Z) = \sum_s z_s - a$. *Then*

$$\delta^*(q|Z) + \delta(q|Q) = a + \begin{cases} 0 & \text{if } q = 1_S \\ \infty & \text{otherwise} \end{cases}$$

where 1_S *denotes a vector of ones.*

Example 6. *Let* $\delta^Q(z|Z) = ||z|| - a$. *Then*

$$\delta^*(q|Z) + \delta(q|Q) = a + \begin{cases} 0 & \text{if } ||q||_* \leq 1 \\ \infty & \text{otherwise} \end{cases}$$

where

$$||q||_* \equiv \sup_x \{q'z : ||z|| \leq 1\}.$$

Example 7. *Let* $\delta^Q(z|Z) = a - \max_{\mu,\lambda\in\Delta_J}\left\{\mu : \mu z \leq \sum_j \lambda_j z^j\right\}$ *where* $\Delta_J \equiv \left\{x \in \mathbb{R}^S_+ : \sum_s x_s = 1\right\}$ *denotes the unit simplex in* $\mathbb{R}^S$ *and* $z^j \in \mathbb{R}^S,\ j = 1,\ldots,J$. *Then*

$$\delta^*(q|Z) + \delta(q|Q) = \sup_{z,\mu,\lambda\in\Delta_J}\left\{q'z + \mu : \mu z \leq \sum_j \lambda_j z^j\right\} - a$$

Example 8. *Let* $\delta^Q(z|Z) = \delta^*(z|X^*) - a$ *for nonempty closed convex* $X^* \in \mathbb{R}^{S*}$. *Then* $\delta^*(q|Z) + \delta(q|Q) = a + \delta(q|X^*)$.

Example 9. *Let Let* $\delta^Q(z|Z) = \delta(z|X) - a$ *for nonempty closed convex* $X \in \mathbb{R}^S$. *Then* $\delta^*(q|Z) + \delta(q|Q) = a + \delta^*(q|X)$

Example 10. *Let* $f^n : \mathbb{R}^S \to \bar{\mathbb{R}},\ n = 1,\ldots,N$ *be proper closed convex and*

$$\delta^Q(z|Z) = \max\left\{f^1,\ldots,f^N\right\}.$$

Then $\delta^Q(z|Z)$ *is closed convex (for example, Rockafellar (1970, Theorem 5.5)) and*

$$\begin{aligned}\delta^*(q|Z) + \delta(q|Q) &= \sup_z\left\{q'z - \max\left\{f^1(z),\ldots,f^N\{z\}\right\}\right\}\\ &= \sup_z \min_n \{q'z - f^n(z)\}\\ &= \min_n \{f^{n*}(q)\}\end{aligned}$$

Composition Results

As Examples 5-10 illustrate, constructing the conjugates for simple choices of δ^Q is straightforward. The same is true for simple choices of Z and Q. But practical instances may require more complex settings and more complex manipulations. In such instances, one strategy is to follow McFadden (1978) and solve parts of the conjugacy correspondence for which the underlying dual relationships are tractable and then use Proposition 2. The next proposition presents composition rules for some common convex forms.

Proposition 4. *a) Let* $f^n : \mathbb{R}^S \to \bar{\mathbb{R}},\ n = 1,\ldots,N$ *be proper closed convex and*

$$\delta^Q(z|Z) = \sum_n f^n(z).$$

Then

$$\delta^*(q|Z)+\delta(q|Q)=\inf_{q^1,\dots,q^N}\left\{\sum_n f^{n*}(q^n):\sum_n q^n=q,\ q^n\in\mathbb{R}^{S*},n=1,\dots,N\right\}$$

b) Let $f^n:\mathbb{R}^S\to\bar{\mathbb{R}},\ n=1,\dots,N$ *be proper closed convex and*

$$\delta^Q(z|Z)=\inf_{z^1,\dots,z^N}\left\{\sum_n f^n(z^n):\sum_n z^n=z,\ z^n\in\mathbb{R}^S,n=1,\dots,N\right\}.$$

Then

$$\delta^*(q|Z)+\delta(q|Q)=\sum_n f^{n*}(q)$$

c) Let $\delta^Q(z|Z)\equiv\max_{j=1,\dots,J}\{z'\bar{q}^j-b_j\}$ *with* $\bar{q}^j\in C^*$ *and* $b_j\in\mathbb{R}$ $j=1,\dots,J$*. Then (co* $\{\cdot\}$ *denotes the convex hull of* $\{\cdot\}$ *in the following)*

$$\delta^*(q|Z)+\delta(q|Q)=\min_\lambda\left\{\sum_{j=1}^J\lambda_j b_j:\lambda\in\Delta_J\ \ q=\sum_{j=1}^J\lambda_j\bar{q}^j\right\},$$

for all $q\in co\left\{\bar{q}^1,\dots,\bar{q}^J\right\}=dom\ \delta^{Q*}$

d) Let $z^j\in\mathbb{R}^S,\ j=1,\dots,J$ *and*

$$\delta^*(q|Z)=\begin{cases}\max_{j=1,\dots,J}\{q'z^j\} & \text{if } q\in C^*\\ \infty & \text{otherwise.}\end{cases}$$

Then for $z\in dom\ \delta^Q$*:*

$$\delta^Q(z|Z)=\inf_{d,\lambda}\left\{\delta^*(d|Q):\lambda\in\Delta_J,z-d\in\sum_{j=1}^J\lambda_j z^j+C\right\}$$

e) Let $Z=C$*. Then*

$$\delta^Q(z|Z)=\inf_{z^o}\{\delta^*(z-z^o|Q):z^o\in C\}$$

f) Let $Z=Z^1\cap Z^2\cap\cdots\cap Z^K$ *with each* $Z^k\subset\mathbb{R}^S$ *nonempty closed convex and* $\bigcap_k ri\ Z^k\neq\emptyset$*. Then*

$$\delta^Q(z|Z)=\sup_{q^1,\dots,q^K}\left\{\sum_k\delta^*\left(q^k|z-Z^k\right)-\delta\left(\sum_k q^k|Q\right)\right\}$$

g) Let $\delta^*(q|Z)=\sum_k\delta^*\left(q|Z^k\right)$ *with* $Z^k\in\mathbb{R}^S,\ k=1\dots,K$ *closed convex. Then*

$$\delta^Q(z|Z)=\inf_{z^1,\dots,z^K}\left\{\sum\delta\left(z^k|Z^k\right)+\delta^*\left(z-\sum_k z^k|Q\right)\right\}.$$

h) Let $Q = Q^1 \cap Q^2 \cap \cdots \cap Q^K$ *with each* $Q^k \subset \mathbb{R}^{S*}$ *nonempty closed convex and* $\bigcap_k ri\ Q^k \neq \emptyset$*. Then*

$$\delta^Q(z|Z) = \inf_{d^1,\ldots,d^K} \left\{ \delta\left(z - \sum_k d^k | Z \right) + \sum_k \delta^*\left(d^k|Q^k\right) \right\}$$

.

i) Let $\delta^*(d|Q) = \sum_k \delta^*\left(d|Q^k\right)$ *with* $Q^k \in \mathbb{R}^{S*},\ k = 1\ldots, K$ *closed convex. Then*

$$\delta^Q(z|Z) = \inf_d \left\{ \delta(z - d|Z) + \sum_k \delta^*\left(d|Q^k\right) \right\}.$$

j) Let $q^j \in \mathbb{R}^{S*},\ j = 1, \ldots, J$ *and*

$$\delta^*(d|Q) = \begin{cases} \max_{j=1,\ldots,J} \{d' q^j\} & \text{if } d \in Q^*_\infty \\ \infty & \text{otherwise.} \end{cases}$$

Then for $z \in dom\ \delta^Q$

$$\delta^Q(z|Z) = \sup_{q,\lambda \in \Delta_J} \left\{ \delta^*(q|z - Z) : q \in \sum_{j=1}^J \lambda_j q^j + Q_\infty \right\}.$$

k) Let $Q = K^*$ *with* $K^* \subset C^*$ *a nonempty closed convex cone.*

$$\delta^Q(z|Z) \quad = \quad \sup\left\{ \delta^*(q|z - Z) : q \in K^* \right\}.$$

l) Let $Q = \{\tilde{q}\}$*. Then*

$$\delta^Q(z|Z) = \delta^*(\tilde{q}|z - Z).$$

m) Let $0 \in Q$*. Then*

$$\delta^Q(z|Z) = \inf_d \left\{ \delta(z - d|Z) + \gamma(d|Q_*) \right\}$$

with $Q_* \equiv \left\{ x \in \mathbb{R}^S : \delta^*(x|Q) \leq 1 \right\} \mathbb{R}^S$ *closed convex.*

Proof: See Appendix.

Discussion of Composition Rules

Antecedents exist for a number of the results in Proposition 4. We highlight some in the following discussion. Note, however, that different restrictions on Z and Q can yield the same δ^Q. The jointness inherent in $\delta^*(q|Z) + \delta(q|Q)$ can manifest itself in identification issues associated with isolating the precise structures that generate δ^Q. Further evidence of the inherent identification problem comes from the Färe et al. (2019), Färe and Zelenyuk (2020), and Zelenyuk and Zhao (2024) demonstrations of alternative strategies for generating Russell-type, slack, and other measures.

Parts a) through c) of Proposition 4 develop a set of calculus rules for the conjugacy operation $\delta^Q(z|Z) \overset{*}{\to} \delta^*(q|Z) + \delta(q|Q)$ for some familiar convex forms. Cases a) and b) correspond to addition and infimal convolution.

Case c) uses the observation that a closed convex function is the pointwise supremum of the affine functions that it majorizes to construct a closed convex function on $\mathbb{R}^S$ from elements of $\mathbb{R}^{S*}$ and $\mathbb{R}$. It supports computation of an inefficiency measure in instances where prior knowledge or observation contain information on dual variates q. For example, let $(\bar{z}^j, \bar{q}^j) \in \mathbb{R}^S \times \mathbb{R}^{S*}$, $j = 1, \ldots, J$ represent J observations on z and q. Then setting $b_j = \bar{q}^{j\prime}\bar{z}^j$ gives

$$\delta^Q(z|Z) = \max_{j=1,\ldots,J} \left\{\bar{q}^{j\prime}\left(z - \bar{z}^j\right)\right\}$$

and

$$\delta^*(q|Z) + \delta(q|Q) = \min_{\lambda} \left\{ \sum_{j=1}^{J} \lambda_j \bar{q}^{j\prime}\bar{z}^j : \lambda \in \Delta_J \ \ q = \sum_{j=1}^{J} \lambda_j \bar{q}^j \right\},$$

as the dual conjugates (Hanoch and Rothschild, 1972). Both are computable using linear programming techniques. Here *dom* $\delta^*(q|Z) + \delta(q|Q) = co\left\{\bar{q}^1, \ldots, \bar{q}^J\right\}$.

Parts d)-m) give calculus rules for the conjugacy operation $\delta^Q(z|Z) \overset{*}{\leftarrow} \delta^*(q|Z) + \delta(q|Q)$ for different Z and Q.

Case d) is the most familiar. It generalizes the Banker et al. (1984) variable-returns model to accommodate netputs and a general Z_∞.[12] The resulting conic program isolates endogenous projections of z falling in $co\left\{z^1, \ldots, z^J\right\} + C$ that support the boundary of Q. When coupled with d), different choices for Q generate different classes of existing inefficiency measures.

[12]The related literature using the canonical DEA model is vast and impractial to cite parsimoniously.

As a first example, let $Q = \{\tilde{q}\}$ with $\tilde{q} \in C^*$. Then

$$\begin{aligned}\delta^Q(z|Z) &= \min_{\lambda}\left\{\tilde{q}\left(z - \sum_j \lambda_j z^j\right) : \lambda \in \Delta_J\right\} \\ &= \delta^*\left(\tilde{q}|z - co\left\{z^1, \ldots, z^J\right\}\right),\end{aligned}$$

which is the $\tilde{q}$-Nerlovian inefficiency for z for $co\left\{z^1, \ldots, z^J\right\}$. Special cases include netput versions of Färe and Lovell's (1978) *Russell* efficiency measure, the Charnes et al. (1985) additive efficiency measure, and the weighted-additive efficiency measure (Lovell and Pastor 1995, Cooper et al. 2011, Aparicio, Pastor, and Vidal 2016, Chambers 2023). Each reduces to a support function for the Minkowski set difference $z - Z$ which is the dual conjugate of the indicator function for that set difference.

Now let $Q = \{q \in C^* : q'z = 1\}$. Then

$$\begin{aligned}\delta^Q(z|Z) &= \inf_{\lambda}\left\{\sup_q\left\{q'\left(z - \sum_j \lambda_j z^j\right) : q'z = 1\right\} : \lambda \in \Delta_J\right\} \\ &= \inf_{\lambda}\sup_q\left\{q'\left(z - \sum_j \lambda_j z^j\right) : q'z = 1, \lambda \in \Delta_J\right\} \\ &= \sup_q\left\{q'z - \delta^*\left(q|co\left\{z^1, ..., z^J\right\}\right) : q'z = 1\right\};\end{aligned}$$

The result is a transformation of a netput version of the Farrell (1957) inefficiency measure.

Applied inefficiency analysts often segregate inputs and outputs. Now partition z as $z' = (\tilde{z}', \hat{z}')$ with $\tilde{z}' \in \mathbb{R}^M$ and $\hat{z} \in \mathbb{R}^{S-M}$, partition q conformably, and define $Q = \{q \in C^* : \tilde{q}'\tilde{z} = 1\}$. Then

$$\delta^Q(z|Z) = 1 + \sup_{q \in Z^*_\infty}\left\{\hat{q}'\hat{z} - \delta^*\left(q|co\left\{z^1, \ldots, z^J\right\}\right) : \tilde{q}'\tilde{z} = 1\right\},$$

defines a transformation of the generalization of the Charnes et al. (1978) and Banker et al. (1984) inefficiency measures that accommodates segregating inputs and outputs and other partitionings of netputs needed to accommodate the presence of quasi-fixed vs.variable inputs, desirable, undesirable outputs, and other departures from the canonical set up.

Leaving that example and turning to case e), we see that the example of conical Z crystallizes the role that Q plays in determining $\delta^Q(z|Z)$. Figure 1 illustrates. There, the left-hand panel illustrates a conical Z and the right-hand panel *dom* $\delta^*(q|Z)$. Constant returns ensures that

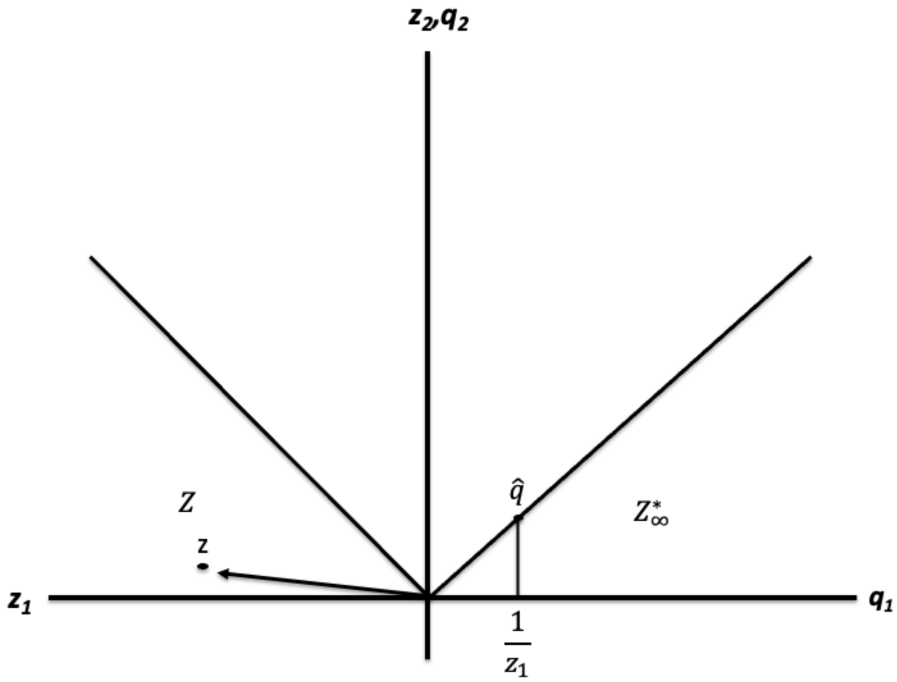
$$\delta^*(q|Z) = \begin{cases} 0 & \text{if } q \in Z_\infty^* \\ \infty & \text{otherwise.} \end{cases}$$

For the depicted $z \in Z$, $q'z \leq 0$ for all $q \in Z_\infty^*$, thus

$$\sup\{q'z - \delta^*(q|Z) : q \in Z_\infty^*\} = 0.$$

Let $Q = \{q \in Z_\infty^* | q_1 z_1 = 1\}$, depicted by the closed line segment connecting q and $\left(\frac{1}{z_1}, 0\right)$. Then $\delta^Q(z|Z) = \hat{q}'z < 0$, where the inequality is confirmed by noting that the angle formed by z and $\hat{q}$ is obtuse.

Figure 1: Figure 1: Constant Returns and Q



Cases f) and g) characterize Z exhibiting different forms of decomposability or separability. Taking each Q^k to be a closed half space gives polyhedral Z as a special case. Therefore, case d) is a special case of f). Case f) also includes what Frisch (1965) calls *multi-dimensional assortment*, where Z is composed of subsets of $\mathbb{R}^S$ that represent sub-processes or stages in production. Such structures have played an important role in extending canonical DEA models to accommodate joint production of desirable and undesirable outputs (for example, Ayres and Kneese 1969, Førsund 1998, Coelli, Lauwers, and Huylenbroeck 2007, Podinovski and Kuosmanen 2011, Murty, Russell, and Levkoff 2012, Chen and Delmas 2012, Chambers, Serra, and Lansink 2014, Baležntis, Blancard, Shen, and Štreimikiė 2021, Kao and Hwang 2021, Murty and Russell 2022).

To illustrate, partition z as $z = (z_d, z_u, z_i)'$ and let $Z = Z^D \cap Z^U$ where

$$
\begin{aligned}
Z^D &= \left\{ z \in \mathbb{R}^S : f(z_i, z_d) = 0 \right\}, \ \textit{and} \\
Z^U &= \left\{ z \in \mathbb{R}^S : g(z_i, z_u) = 0 \right\}.
\end{aligned} \tag{24}
$$

Here subscript d denotes desirable outputs, u denotes undesirable outputs or byproducts, i denotes inputs and f and g represent numeric functions. Z^D represents the desirable production process and Z^U the clean-up or abatement process. Kohli (1983) classifies the form in (24) as "output-price nonjoint". It depicts a production process that "cracks" a fixed input-bundle, z_i, into separate bundles of desirable and undesirable outputs. The output-price nonjoint specification is common in pollution-abatement studies, where desirable and undesirable outputs are often assumed to be produced in fixed proportions (Murty and Russell 2022). Case f) also includes the "event-contingent technologies" (Chambers and Quiggin 2000, O'Donnell and Griffiths 2006, Chambers, Hailu, and Quiggin 2011, Chambers, Serra, and Stefanou 2015, Serra, Chambers, and Lansink 2014, Chambers et al. 2014) used in studies of inefficiency in the presence of uncertainty.

Case g) models Z as the infimal convolution of J distinct sub-processes

$$
\delta\left(z|Z\right) = \inf_{z^1,\ldots,z^J} \left\{ \sum_j \delta\left(z^j|Z^j\right) : \sum_j z^j = z \right\}
$$

It characterizes, for example, decisionmakers or enterprises that operate across separate plants or locations. It also generalizes familiar notions of input and output nonjoint technologies to netput-nonjointness that allows netputs to be freely allocated across the sub-processes.

Cases h), i), and j) model situations where the normalization associated with (12) can be decomposed into subsets of normalizing restrictions. Case h) treats Q as the intersection of a finite series of convex sets. Restricting each Q^k to be polyhedral, cases h) and j) generalize broad classes of DEA-based measures including Charnes et al. (1978), Banker et al. (1984), directional distance functions, weighted average measures, various Russell measures, Ray (2007), and the DEA measures studied by Pastor et al. (2012). To illustrate, let $Q = Q^1 \cap Q^2 \cap Z^*_\infty$ with $Q^1 = \{q : q'g^1 \leq 1\}$ and $Q^2 = \{q : q'g^2 \leq 1\}$ with $g^1, g^2 \in \mathbb{R}^S$. Then

$$\delta^Q(z|Z) = \sup_{q \in Z^*_\infty} \left\{ q'z - \delta^*(q|Z) : q'g^1 \leq 1, q'g^2 \leq 1 \right\},$$

which generalizes Ray's (2007) Overall (Shadow Profit) inefficiency measure (Ray 2007, Aparicio et al. 2013, and Ray and Yang 2024) to accommodate arbitrary Z and Z_∞

Cases k) and l) are straightforward consequences of (12) and need little elaboration.

Case m) shows that, when the origin belongs to Q, the class of finite Paretian inefficiency measures specializes to the class of "minimal gauge functions"

$$\delta^Q(z|Z) = \inf_d \{\gamma(d|Q_*) : z - d \in Z\}.$$

The class of "minimal-norm inefficiency measures", in turn, is the subset of the minimal-gauge inefficiency measures for bounded Q symmetric about zero. A *norm*, $\rho : \mathbb{R}^S \to \bar{\mathbb{R}}$, satisfies: a) $\rho(x) \geq 0$ for all $x \in \mathbb{R}^S$, b) $\rho(\alpha x) = |\alpha| \rho(x)$ for $\alpha \in \mathbb{R}$; and c) subadditivity (the triangle inequality). Let Q be closed convex bounded and symmetric about 0. Then $\gamma(q|Q)$ defines a norm for $\mathbb{R}^{S*}$, $\rho_* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, whose polar form

$$\begin{aligned} \rho(d) &= \sup_q \{q'd : \rho_*(q) \leq 1\} \\ &= \sup_q \{q'd : \gamma(q|Q) \leq 1\} \\ &= \delta^*(d|Q) \end{aligned}$$

is a norm for $\mathbb{R}^S$ with unit ball $\left\{d \in \mathbb{R}^S : \delta^*(d|Q) \leq 1\right\}$ (see, for example, Rockafellar 1970, Theorem 15.2). Special cases of ρ include the L_p norms used to define the Hölder distance functions studied by Briec (1998) and Briec and Lesourd (1999). Taking ρ_* to be the Euclidean norm $|| \cdot ||$ so that $Q = \left\{q \in \mathbb{R}^{S*} : ||q|| \leq 1\right\}$ gives

$$\delta^Q(z|Z) = \inf_d \{||d|| : z - d \in Z\}$$

as the minimum Euclidean distance to translate z while maintaining that it belongs to Z.

Mind the Gap: Exogenous q and Measuring Dual Inefficiency

Oftentimes, exogenous information on dual variates, say q^o, is available. Coupled with knowledge of z and Z, that information permits calculation of the q^o-Nerlovian inefficiency for z. Because $\delta^Q(z|Z)$ maximizes Nerlovian inefficiency over Q, it can diverge from $\delta^*(q^o|z-Z)$. A tradition that traces to Farrell (1957) uses that observation to decompose q^o-Nerlovian efficiency for z for exogenous q^o into a *technical inefficiency* component and a *dual inefficiency* component.[13]

Fenchel's Inequality

Fenchel's Inequality (4) offers a natural means to examine efficiency decompositions. Applying it to δ^Q gives the recycled version of (14):

$$\delta^Q(z|Z) \geq \delta^*(q|z-Z) - \delta(q|Q) \tag{25}$$

for all z, q. Using (25), we define $\varphi^Q : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}_+$ as

$$\varphi^Q(q, z|Z) \equiv \begin{cases} \delta^Q(z|Q) + \delta^*(q|Z) - q'z & \text{if } q \in Q \\ \infty & \text{otherwise.} \end{cases}$$

φ^Q is closed convex as a function of z, closed convex as a function of q, closes any duality gap implied by a strict inequality in (25), and *for* $q \in Q$ measures the difference between $z's$ Pareto inefficiency and its Nerlovian inefficiency. Thus,

$$\delta^*(q|z-Z) = \delta^Q(z|Z) - \varphi^Q(q, z|Z), \tag{26}$$

for $q \in Q$ and all z.

[13] More common terms for dual inefficiency are either *allocative* or *price* inefficiency. For example, Afriat (1972, p. 582), who worked in a cost context, partitioned the "total gap...as a part due to the inefficiency of output with respect to the input, and...misplaced allocation of cost over inputs". When $\mathbb{R}^{S*}$ represents price, "virtual-price", or "shadow-price" space, the allocative or price inefficiency interpretation is natural. The dual terminology covers both and is more descriptive of its mathematical structure. Thus, it also applies to non-price settings.

Expression (26) invites the interpretation of $\varphi^Q(q^o, z)$ as the *dual inefficiency* component of Nerlovian inefficiency at q^o and of $\delta^Q(z|Z)$ as the *technical inefficiency.* Interpretive issues intrude, however, for $q^o \notin Q$ because $\varphi^Q(q^o, z|Z)$ then becomes arbitrarily large. That implies an infinitely large dual inefficiency conveying the intuition that relative to q^o a decisionmaker at z makes arbitrarily large bad decisions, when in truth $\delta^*(q^o|z - Z)$ and $\delta^Q(z|Z)$ are not commensurable. This intuition founders because the one-to-one relationship between $\delta^Q(z|Z)$ and $\delta^*(q|z - Z)$ becomes noninformative when $q \notin Q$.

Recent work on *axiomatic inefficiency measurement* provides another lens through which to analyze this aspect of φ^Q. The axiomatic approach evaluates different inefficiency measures by their ability to satisfy certain axioms (see, for example, Russell and Schworm 2009 and 2017). Usually, axioms are imposed on the proposed technical inefficiency measure, our δ^Q. Aparicio, Zofio, and Pastor (2023) argue that technical inefficiency measures should also *be judged on the behavior of their associated dual inefficiency measures.* They propose that a candidate dual inefficiency measure, $I : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}$, satisfy an *Essential Property* summarized, in our notation, as[14]

Property 1. *If $z \in \partial\delta^*(q|Z)$, then $I(q, z) = 0$.*

Aparicio et al. (2023) use examples to show that the Russell Graph Measure, the Enhanced Russell Graph, the Additive, and the Weighted Additive measures do not satisfy Property 1. Because these measures correspond to cases rationalized by a singleton $Q = \{\tilde{q}\}$, $\varphi^Q(q, z|Z) = \infty$ for $q \neq \tilde{q}$ for them. Failure of a measure to satsify Property 1 translates in our setting into noncommensurability between Nerlovian and Paretian inefficiency.

More generally, as a consequence of φ^Q's convexity and Proposition 2, we have:

Corollary 1. *Let $\delta^Q(z|Z)$ be finite. Then φ^Q is zero-minimal if and only if*

$$q \in \partial\delta^Q(z|Z) \Leftrightarrow z \in \partial\delta^*(q|Z) + \partial\delta(q|Q)$$

The Russell measures, the Additive measure, and the Weighted Additive cannot satisfy Corollary 1 for arbitrary q when $Q = \{\tilde{q}\}$ because then $\partial\delta(q|Q) = \emptyset$ for $q \neq \tilde{q}$. Figure 2

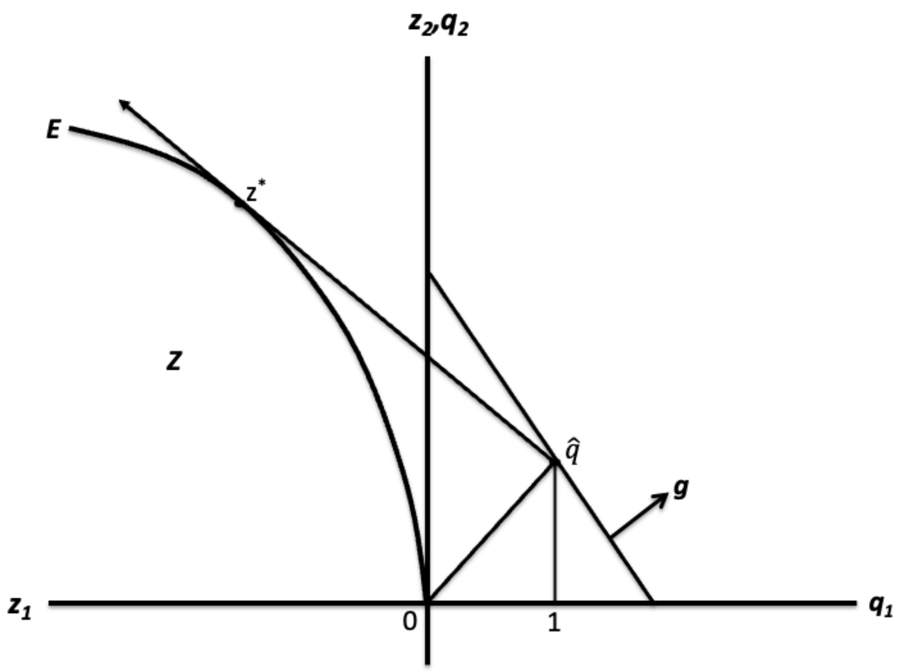
[14] Aparicio et al. (2023) segregate inputs and outputs and discuss separate input and output-oriented versions of their Essential Property for both ratio-based and difference-based measures. We only consider difference-based measures for netputs and leave the natural extensions to the interested reader.

illustrates the phenomenon. We assume that $Z_\infty = \mathbb{R}^2_-$ and the efficient set is the curve labelled $0E$. First, let Q be the closed line segment connecting $(1,0)$ and $\hat{q}$. Then the set of efficient projections is restricted to points falling below z^* on OE. In particular, points on the arc beyond z^* on OE cannot be efficient projections.

Now, let $Q = \{q : q'g = 1\}$ (a directional distance function) for the g illustrated there. Q now encompasses all directions in C^* so that no point on OE is excluded from $P^Q(q|Q)$. φ^Q derived from these measures satisfies Property 1.

The technical issue is that certain choices for Q can restrict the range of $P^Q(z|Z)$ enough to make the link between $\delta^Q(z|Z)$ and $\delta^*(q|z-Z)$ implied by Fenchel's Inequality noninformative. Or more simply, as Chambers (2023) observed, choosing a Nerlovian inefficiency measure for an arbitrary $\tilde{q}$ to measure technical inefficiency, when information on exogenous q is available, distorts the distinction between technical and dual inefficiency.

Figure 2: Figure 2: Essential Property Fails



We note that our use of Fenchel Inequality's, (25), differs from related discussions in some studies (see, for example, Färe and Grosskopf 1997, Chambers et al. 1998, Chambers and Färe 2004, Cooper et al. 2011, Zofio, Pastor, and Aparicio 2013, Aparicio, Borras, Pastor, Vidal 2013, Petersen 2018). The difference is more semantic than substantive, but it merits clarification. Ours follows established terminology in convex analysis, but some inefficiency studies use different naming conventions.

As a first example, let $Q = \left\{q \in \mathbb{R}^{S*} : q'g = 1\right\}$ with $g \in Z_{\infty}$. Then

$$\begin{aligned} \delta^{Q}(z|Z) &= \sup_{q} \{q'z - \delta^{*}(q|Z) - \delta(q|Q)\} \\ &= \sup_{q} \{q'z - \delta^{*}(q|Z) : q'g = 1\} \\ &= \sup_{q} \left\{ \frac{q}{q'g}' z - \delta^{*}\left(\frac{q}{q'g}|Z\right)\right\}, \end{aligned} \tag{27}$$

where we use the sublinearity of δ^{*}. Consequently, reasoning similar to (4) gives:

$$\delta^{Q}(z|Z) \geq \frac{q}{q'g}' z - \delta^{*}\left(\frac{q}{q'g}|Z\right) \text{ for all } q, z. \tag{28}$$

Different writers have named (28) and its analogues differently, including the *Luenberger Inequality*, the *Fenchel-Mähler Inequality*, the *generalized Fenchel-Young Inequality*, among others.[15] Each communicates the same message. The inefficiency measure for z is an upper bound for a $q-$ Nerlovian inefficiency. Let $\hat{q}$ solve (27), then

$$\left(\frac{q}{\hat{q}'g} - \frac{q}{q'g}\right)' z + \delta^{*}\left(\frac{q}{q'g}|Z\right) - \delta^{*}\left(\frac{\hat{q}}{\hat{q}'g}|Z\right)$$

closes any gap in (28) and gives a "real" (in units of the directional bundle g) of the length of that gap. For (27) picking the numeraire bundle, g, also determines the efficient direction and efficient projection. $D^{Q}(z|Z)$ is a scalar multiple of g and $P^{Q}(z|Z)$ is the projection of z onto $EffZ$ in the direction g for all z.

More generally, however, $D^{Q}(z|Z)$ and $P^{Q}(z|Z)$ can vary with z. Another example

[15]For $X \subset \mathbb{R}^{S}$ closed convex and containing the origin, its gauge and support functions are polar to one another. Hence,

$$\delta^{*}(q|X) = \sup\{q'x : \gamma(x|X) \leq 1\},$$

whence $\delta^{*}(q|X)\gamma(x|X) \geq 1$, which manifests Mähler's Inequality.

illustrates. Let $Q = \left\{ q \in \mathbb{R}^{S*} : \sum_{s \in 1,\ldots,S} |q_s| \leq 1 \right\}$. Then

$$\begin{aligned} \delta^Q (z|Z) &= \sup_q \left\{ q'z - \delta^* (q|Z) : \sum_{s \in 1,\ldots,S} |q_s| \leq 1 \right\} \\ &= \sup_q \left\{ \frac{q}{\sum_{s \in 1,\ldots,S} |q_s|}' z - \delta^* \left(\frac{q}{\sum_{s \in 1,\ldots,S} |q_s|} | Z \right) \right\} \\ &= \inf_d \left\{ \max_{s \in 1,\ldots,S} \{|d_s|\} : z - d \in Z \right\}. \end{aligned} \tag{29}$$

The third equality follows by (16) since now $\delta^* (d|Q) = \max_{s \in 1,\ldots,S} \{|d_s|\}$. Thus, one can evocatively write

$$\delta^Q (z|Z) \geq \frac{q}{\sum_{s \in 1,\ldots,S} |q_s|}' z - \delta^* \left(\frac{q}{\sum_{s \in 1,\ldots,S} |q_s|} | Z \right) \text{ for all } q, z. \tag{30}$$

Without loss of generality, let the optimizer for (29) be d_1. One can now use subtraction in (30) to derive a dual inefficiency measure. But for the real units to comparable, the right-hand side of (30) needs to evaluated in units of z_1. Because the the element of z that optimizes (29) can vary with z, the numeraire will vary with z.

Expressions (25), (28), and (30) all convey similar information. Expression (25) gives a dual inefficiency measure that closes the "gap" by simple subtraction, but it operates in what an economist calls "nominal" (albeit restricted to lie in Q) rather than "real" units. Expression (28) also gives a dual inefficiency measure by simple subtraction that is expressed in real terms, but it restricts $D^Q (z|Z)$. Expression (30) yields a real dual inefficiency measure by simple subtraction, but it requires identification of different numeraire for each application. Hence, dual inefficiency is not directly comparable across different z.

Conjugacy Correspondences for Dual Inefficiency

Because it is closed convex in q and closed convex in z, φ^Q is what Rockefellar (1970, Section 29) refers to as a bi-function. That observation implies that φ^Q has "a life of its own" as part of at least two well-defined conjugacy correspondences.

In particular, when viewed from a Lagrangean perspective

$$\varphi^{Q}(q,z|Z)=\delta^{Q}(z|Z)+\delta^{*}(q|Z)+\delta(q|Q)-q'z$$

admits two parallel interpretations: one as the Lagrangean function for a closed convex program for minimizing $\delta^{Q}(z|Z)+\delta^{*}(q|Z)+\delta(q|Q)$ over q; and the other as the Lagrangean function for a closed convex program for minimizing $\delta^{Q}(z|Z)+\delta^{*}(q|Z)+\delta(q|Q)$ over z. In the first, z plays the role of a Lagrange multiplier and q plays that role in the second.

Define $\varphi^{Q**}:\mathbb{R}^{S}\times\mathbb{R}^{S}\rightarrow\bar{\mathbb{R}}$

$$\begin{aligned}\varphi^{Q**}(\bar{z},z|Z) &\equiv \sup_{q\in\mathbb{R}^{S*}}\left\{q'\bar{z}-\varphi^{Q}(q,z|Z)\right\} \\ &= \delta^{Q}(\bar{z}+z|Z)-\delta^{Q}(z|Z).\end{aligned} \tag{31}$$

Expression (31) defines the (partial) conjugate of $\varphi^{Q}(q,z|Z)$ treated as a closed convex function of q. The second equality follows by (12) and Proposition 2. We have:

Proposition 5. $\varphi^{Q}(q,z|Z)\overset{*}{\longleftrightarrow}\delta^{Q}(\bar{z}+z|Z)-\delta^{Q}(z|Z)$

$$\varphi^{Q}(q,z|Z)+\delta^{Q}(\bar{z}+z|Z)-\delta^{Q}(z|Z)\geq q'\bar{z}\ \ \forall q,\bar{z}\quad \textit{(Fenchel's Inequality)}$$

$$\hat{z}\in\partial_{q}\varphi^{Q}(q,z|Z)\Leftrightarrow q\in\partial_{\hat{z}}\delta^{Q}(\hat{z}+z|Z|)\Leftrightarrow\varphi^{Q}(q,z|Z)+\delta^{Q}(\bar{z}+z|Z)-\delta^{Q}(z|Z)=q'\hat{z}$$

Symmetrically, we define $\varphi^{Q*}:\mathbb{R}^{S*}\times\mathbb{R}^{S*}\rightarrow\bar{\mathbb{R}}$ as the (partial) conjugate of φ^{Q} treated as a closed convex function of z:

$$\begin{aligned}\varphi^{Q*}(q,\bar{q}|Z) &\equiv \sup_{z\in\mathbb{R}^{S}}\left\{\bar{q}'z-\varphi^{Q}(q,z|Z)\right\} \\ &= \delta^{*}(\bar{q}+q|Z)+\delta(\bar{q}+q|Q)-\delta^{*}(q|Z)-\delta(q|Q),\end{aligned} \tag{32}$$

where the second equality follows by (12) and Proposition 2. Hence,

Proposition 6. $\varphi^{Q}(q,z|Z)\overset{*}{\longleftrightarrow}\delta^{*}(\bar{q}+q|Z)+\delta(\bar{q}+q|Q)-\delta^{*}(q|Z)-\delta(q|Q)$

$$\varphi^{Q}(q,z|Z)+\delta^{*}(\bar{q}+q|Z)+\delta(\bar{q}+q|Q)-\delta^{*}(q|Z)-\delta(q|Q)\geq\bar{q}'z\ \ \forall\bar{q},z\quad \textit{(Fenchel's Inequality)}$$

$$\begin{aligned}\hat{q}\in\partial_{z}\varphi^{Q}(q,zZ) &\Leftrightarrow z\in\partial_{\hat{q}}\delta^{*}(\hat{q}+q|Z)+\partial_{\hat{q}}\delta(\hat{q}+q|Q) \\ &\Updownarrow \\ \hat{q}'z-\delta^{*}(\hat{q}+q|Z)-\delta(\hat{q}+q|Q) &= \varphi^{Q}(q,z|Z)-\delta^{*}(q|Z)-\delta(q|Q)\end{aligned}$$

The dual-inefficiency measure is conjugate dual to a) a transformation of Paretian inefficiency and b) a transformation of Nerlovian inefficiency. Propositions 5 and 6 are direct consequences of Lemma 1 and Proposition 2. Indeed, they convey the same mathematical information. Their force is that they establish that $\varphi^Q(q, z|Z)$, despite its frequent treatment as a residual, forms a component of a two conjugacy correspondences for two closed convex proper inefficiency measures. So, for example, specification of a $\varphi^Q(q, z|Z)$ that is closed convex in q and closed convex in z implies the existence of a well-behaved $\delta^Q(\bar{z} + q|Z) - \delta^Q(z|Z)$ and *vice versa* without the need for "...difficult constructive arguments" (McFadden 1978). Specification of a closed convex $\delta^Q(z|Z)$ implies the existence of a conjugate $\varphi^Q(q, z|Z)$ interpretable as a dual inefficiency measure. Parallel logic applies to $\delta^*(q|Z)$ and $\varphi^Q(q, z|Z)$. In short, if one can specify a meaningful $\delta^Q(z|Z)$, one can measure inefficiency and decompose it meaningfully without resorting to solving (12).

The introduction of Luenberger's (1992) measures into the inefficiency discussion emphasized that different directional orientations changed measures. Recognizing that choice matters, many contributions followed offering different perspectives and insights (Chambers et al. 1998; Tone 2001; Chambers and Färe 2004; Ray 2007; Cooper et al. 2011; Pastor et al. 2012; Aparicio et al. 2013, Zofio et al. 2013; Aparicio et al. 2015; Petersen 2018; Aparicio et al. 2023). Our results formalize a folk tradition that percolates throughout this literature: *Dual inefficiency can be made as large or small as one chooses.* We demonstrate it by first choosing $Q = \{\tilde{q}\}$ for $q \neq \tilde{q}$ to make $\varphi^Q(q, z|Z) = \infty$. But then choosing $Q = \{q\}$ gives $\varphi^Q(q, z|Z) = 0$.

How to use that information is unresolved. But we do observe a marked difference between the "axiomatic approach to decision theory" and the "axiomatic approach to inefficiency measurement" suggests one avenue. Both approaches use axioms and closely related mathematical machinery. But where the decision theory uses deductive reasoning from behavioral axioms to deduce functional preference representations, most axiomatic inefficiency analysts follow a Fisherian test approach with axioms viewed as "tests" that candidate measures should pass.[16] Perhaps a closer integration of the two axiomatic approaches is merited.

[16]Hougaard and Keiding (1998) and Chambers and Miller (2014) are exceptions.

3 Concluding Remarks

We pose inefficiency measurement as measuring the distance between an outcome and the efficient frontier for a closed convex set, Z. We define the efficient frontier using a generalized inequality, $\preceq_C$, that is a reflexive, transitive, and asymmetric binary relation, that partially orders $\mathbb{R}^S$, and that permits characterization of the efficient set using dual variates. We define Paretian inefficiency for an outcome as the maximal distance between the outcome and the support function for Z while restricting dual variates to fall in a closed convex set. That formulation generalizes, while integrating into a common mathematical formulation, broad classes of inefficiency-measurement strategies. The result is a “minimal-support” inefficiency measure with characteristics that resemble the Nirnberg (1961) minimal-norm duality. We show that the resulting Paretian inefficiency measure is proper closed convex and conjugate dual to a restricted Nerlovian ineffiency measure that is proper closed convex. We use that conjugate duality to construct classes of dual composition rules for varying restrictions on Z and dual-variate normalization.

We use the Paretian inefficiency measure to decompose measured Nerlovian inefficiency into a technical-inefficiency measure and a dual-inefficiency measure. The dual-inefficiency measure is a closed convex bi-function that satisifies two dual conjugacies: one with a differenced Paretian inefficiency measure and another with a differenced Nerlovian inefficiency measure. We use the dual-inefficiency measure to investigate recent concerns raised about the appropriateness of Nerlovian inefficiency decompositions.

In defining an inefficiency measure, we follow a path tread earlier by Debreu (1951), Farrell (1957), Afriat (1972), and Charnes et al. (1978) that treats inefficiency measurement as evaluating an outcome in the most favorable dual terms. That setting encompasses broad classes of existing inefficiency measures. But other approaches exist (for example, Pastor (et al. 2007) and Färe et al. (2019)) and their mathematical formulation appears different from ours. Nevertheless these studies often yield measures similar to ours and other authors following our generic approach suggesting that formal identification of the ideal structural formulation for inefficiency measurement awaits further development.

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References

Afriat, S. ”Efficiency Estimation of Production Functions.” International Economic Review 13 (1972): 568-98.

Aliprantis, C. and K. Border. Infinite Dimensional Analysis: A Hitch-Hiker’s Guide (3rd edition). Berlin: Springer, 2007.

Aparicio, J., J. T. Pastor, and F. Vidal. ”The Weighted Additive Distance Function.” European Journal of Operational Analysis 254 (2016): 338-46.

———. ”The Directional Distance Function and the Translation Invariance Property.” Omega 58 (2016): 1-3.

Aparicio, J. ; Borras, F.; Pastor, J.; Vidal, F. ”Measuring and Decomposing Firm’s Revenue and Cost Efficiency: The Russell Measures Revisited.” International Journal of Production Economics 165 (2015): 19-28

Aparicio, J.; Pastor, J.T.; Ray, S.C. ”An Overall Measure of Technical Inefficiency at the Firm and at the Industry Level: The ‘Lost Profit on Outlay’.” European Journal of Operational Research 226 (2013): 154-62.

Aparicio, J.; Zofio, J., Pastor, J.T. ”Decomposing Economic Inefficiency into Technical and Allocative Components: An Essential Property.” Journal of Optimization Theory and Applications 197 (2023): 98-129.

Aubin, J. P. Ekeland I. Applied Nonlinear Analysis. Mineola, NY: Dover Publications, 2007.

Ayres, R. U. and A. W. Kneese. ”Production, Consumption, and Externalities.” American Economic Review 59 (1969): 282-97.

Baležntis, T., Blancard, S., Shen, Z., and Štreimikinė, D. “Analysis of Environmental Total Factor Productivity Evolution in European Agricultural Sector”. Decision Sciences 52 (2021): 483-511.

Banker, R. D., A. Charnes, and W. W. Cooper. ”Some Models for Estimating Technical and Scale Inefficiencies in Data Envelopment Analysis.” Management Science 30, no. 9 (1984): 1078-92.

Bator, A.; Briec, W. “Distance Function in Some Class of Infinite Dimensional Vector Spaces.”Journal of Optimization Theory and Applications 201(2024): 899-931.

Bertsekas, D. P. Convex Optimization Theory. Belmont MA: Athena Scientific, 2009.

———. Nonlinear Programming. Belmont MA: Athens Scientific, 2016.

Boyd, S.;Vandenberghe, L. Convex Optimization. Cambridge: Cambridge University Press, 2004.

Briec, W.“Hölder Distance Function and Measurement of Technical Efficiency.” Journal of Productivity Analysis 11 (1998): 111-31.

Briec, W., and J. B. Lesourd. ”Metric Distance and Profit Functions: Some Duality Results.” Journal of Optimization Theory and Applications 101 (1999): 15-33.

Byrnes, P.,Färe, R.,Grosskopf, S., and Lovell C.A.K. “The Effect of Unions on Productivity: U.S. Surface Mining of Coal” Management Science 34 (1988): 1037-53.

Chambers, C. P., and A. Miller. ”Inefficiency Measurement.” American Economic Journal: Microeconomics 6 (2014): 79-92.

Chambers, R.G. ”Dual Structures for the Additive DEA Model.” European Journal of Operational Research 307 (2023): 984-99.

———. “Numeraire Choice, Shadow Profit, and Inefficiency Measurement.” European Journal of Operational Research (2024, forthcoming).

Chambers, R. G., Y. Chung, and R. Färe. ”Benefit and Distance Functions.” Journal of Economic Theory 70 (1996): 407-19.

———. ”Profit, Directional Distance Functions, and Nerlovian Efficiency.” Journal of Optimization Theory and Applications 98, no. 2 (1998): 351-64.

Chambers, R.G.; Färe, R. “Additive Decomposition of Profit Efficiency”. Economics Letters 84 (2004): 329-24.

———. “Distance Functions”. Handbook of Production Economics, eds. Subhash Ray, Robert G Chambers, and Subal Kumbhakar: Springer, Singapore 2022.

Chambers, R. G., A. Hailu, and J. Quiggin. ”Event-Specific Data Envelopment Models and Efficiency Analysis.” Australian Journal of Agricultural and Resource Economics 55 (2011):1-17.

Chambers, R. G., T. Serra, and A. Oude Lansink. ”On the Pricing of Undesirable Outputs.” European Review of Agricultural Economics 41 (2014): 485-509.

Chambers, R. G. J. Quiggin. Uncertainty, Production, Choice, and Agency: The State-

Contingent Approach. New York: Cambridge University Press, 2000.

Chambers, R. G.; Serra, T.; Stefanou, S. "Using Ex Ante Output Elicitation to Model State-Contingent Technologies." Journal of Productivity Analysis 43 (2015): 75-83.

Charnes, A., W. W. Cooper, B. Golany, L. Seiford, and J. Stutz. "Foundations of Data Envelopment Analysis for Pareto-Koopmans Efficient Empirical Production Functions." Journal of Econometrics 30 (1985): 91-107.

Charnes, A., W. W. Cooper, and E. Rhodes. "Measuring Efficiency of Decisionmaking Units." European Journal of Operational Research 2 (1978): 429-44.

Chen, C.-M.; Delmas, M.A. "Measuring Eco-Inefficiency: A New Frontier Approach." Operations Research 60 (2012): 1064-79.

Coelli, T.; Lauwers, L.; Huylenbroeck,. "Environmental Efficiency Measurement and the Materials Balance Condition." Journal of Productivity Analysis 28 (2007): 3-12.

Cooper, W. W., J. T. Pastor, J. Aparicio, and F. Borras. "Decomposing Profit Ineffiency in DEA through the Weighted Additive Model." European Journal of Operational Research 212 (2011): 411-16.

Debreu, G. "The Coefficient of Resource Utilization." Econometrica 19, no. 3 (1951): 273-92.

Edgeworth, F. Y. Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences. London: G. Kegan Paul, 1881.

Ehrgott, M. Multicriteria Optimization, 2d. Ed. Heidelberg: Springer Berlin, 2005.

Farrell, M. J. "The Measurement of Productive Efficiency." Journal of the Royal Statistical Society 129A (1957): 253-81.

Färe, R.; Grosskopf, S. "Profit efficiency, Farrell decompositions, and the Mähler Inequality". Economics Letters 57 (1997): 283-87.

Färe, R., Grosskopf, S., Lovell, C.A.K. The Measurement of Efficiency of Production. Boston: Kluwer, 1985.

Färe, R.; Grosskopf, S.; Whitaker, G. "Directional output distance functions:endogenous directions based on exogenous normalization constraints." Journal of Productivity Analysis 40: 267-69

Färe, R., and C. A. K. Lovell. "Measuring the Technical Efficiency of Production." Journal of Economic Theory 19 (1978): 150-62.

Färe, R.; He, X.; Li, S.; Zelenyuk. "A Unifying Framework for Farrell Profit Efficiency Measurement." Operations Research 67 (2019): 183-97.

Färe, R. and V. Zelenyuk. "Profit Efficiency: Generalization, business accounting, and the role of convexity". Economics Letters 196 (2020).

Fenchel, W. Convex Cones, Sets, and Functions. Princeton, 1951.

Hanoch, G.; Rothschild, M. "Testing the Assumptions of Production Theory: A Nonparametric Approach." Journal of Political Economy 80 (1972): 256-75.

Hiriart-Urruty, J. B. and LeMaréchal C. Fundamentals of Convex Analysis. Heidelberg Berlin: Springer Verlag, 2001.

Hougaard, J.L and H.Keiding, On the functional form of an efficiency index; Journal of Productivity Analysis 9 (1998): 103-11.

Isac, G., and A. A. Khan. "Dubovitskii-Milyutin Approach in Set-Valued Optimization." SIAM Journal of Control and Optimization 47, no. 1 (2008): 144-62.

Kao, C.; Hwang, S.-N.. "Measuring the effects of undesirable outputs on the efficiency of production units." European Journal of Operational Research 292 (2021): 996-1003.

Kohli, U. "Nonjoint Technologies." Review of Economic Studies 50 (1983): 209-19.

Kuosmanen. T. "Weak disposability in nonparametric productivity analysis with undesirable outputs." American Journal of Agricultural Economics 84 (2005): 1077-82.

Löhne, A. Vector Optimization with Infimum and Supremum. Heidelberg: Springer, 2011.

Lovell, C. A. K., and J. T. Pastor. "Units Invariant and Translation Invariant Dea Models." Operations Research Letters 18 (1995): 147-51.

Luenberger, D. Optimization by Vector-Space Methods. New York: John Wiley and Sons, 1969.

Luenberger, D. G. "New Optimality Principles for Economic Efficiency and Equilibrium." Journal of Optimization Theory and Applications 75, no. 2 (1992): 221-64.

McFadden, D. "Cost, Revenue, and Profit Functions." Production Economics: A Dual Approach to Theory and Applications. edited by M. Fuss and D.McFadden. Amsterdam: North-Holland, 1978.

Mehdiloo, M. and Podinovski, V. "Selective strong and weak disposability in efficiency analysis." European Journal of Operational Research 276 (2019): 1154-1169.

———."Strong, weak, and Farrell efficient frontiers of technologies satisfying different production assumptions". European Journal of Operational Research 294 (2021): 295-311.

———. "Data envelopment analysis with embedded inputs and outputs" Annals of Operations Research 335 (2024): 293-326.

Minkowski, H. Theorie der konvexen Körper. In Gesammelte Abhandlungen, 131–229. Leipzig/Berlin: Teubner II, 1911.

Moreau, J. "Fonctions Duales Convexes Et Points Proximaux Dans Un Espace Hilbertien." Comptes Rendus de l'Académie des Sciences de Paris A255 (1962): 2897-99.

Murty, S., R. R. Russell, and S. Levkoff. "On Modeling Pollution-Generating Technologies." Journal of Environmental Economics and Management, no. 64 (2012): 117-35.

Murty, S.; Russell, R.R. "Bad Outputs." In Handbook of Production Economics, edited by S. C.; Chambers Ray, R.G.; Kumbhakar, R.: Springer, 2022.

Nemirovski, A. Advances in Convex Optimization: Conic Programming. Georgia Institute of Technology 2007.

Newman, P. "Gauge Functions". New Palgrave Dictionary of Economics. edited by Eatwell, J., Milgate, M., and Newman, P. New York: Macmillan 1987.

Nirenberg, L. Functional Analysis. Lecture Notes inscribed by L. Sibner. NYU, 1961.

O'Donnell, C., R. G. Chambers, and J. Quiggin. "Efficiency Analysis in the Presence of Uncertainty." Journal of Productivity Analysis 33 (2010):1-17.

O'Donnell, C. J., and W. Griffiths. "Estimating State-Contingent Production Frontiers." American Journal of Agricultural Economics 88 (2006): 249-66

Pareto, V. Manuel d'économie Politique. Paris: M. Girard, 1909.

Pastor, J. T.; Lovell, C.A.K.; Aparicio, J. "Families of Linear Efficiency Programs Based on Debreu's Loss Function." Journal of Productivity Analysis 38 (2012): 109-20.

Pastor, J.T.; Ruiz, J.L.; Sirvent, I. "Closest Targets and Minimum Distance to the Pareto-Efficient Frontier." Journal of Productivity Analysis 28 (2007): 209-18.

Petersen, N.C. "Directional Distance Functions in DEA with Optimal Endogenous Directions." Operations Research 66 (2018): 1068-85.

———. "Units of measurement and directional distance functions with optimal endogenous directions in data envelopment analysis". European Journal of Operational Research 282 (2020): 712-28.

Pham, M.D. and Zelenyuk, V. "Weak disposability in nonparamteric production analysis: A new taxonomy of reference technology sets." European Journal of Operational Research 274 (2018): 186-96.

Podinovski, V.V. and Førsund, F. "Differential Characteristics of Efficient Frontiers in Data Envelopment Analysis". Operations Research 58 (2010): 1743-54.

Podinovski, V.V.; Kuosmanen, T. "Modelling weak disposability in data envelopment analysis under relaxed convexity assumptions". European Journal of Operational Research 211 (2011): 577-85

Ray, S. C. Data Envelopment Analysis: Theory and Techniques for Economics and Operations Research. New York: Cambridge University Press, 2004.

———. "Shadow Profit Maximization and a Measure of Overall Inefficiency." Journal of Productivity Analysis 27 (2007): 231-36.

Ray. S.C. and Yang, L. Decomposition of Profit Inefficiency under Alternative Definitions of Technical Inefficiency. Manuscript 2024.

Rockafellar, R. T. Convex Functions and Dual Extremum Problems. Cambridge: PhD Thesis, Harvard University, 1963.

———. Convex Analysis. Princeton: Princeton University Press, 1970.

Rockafellar, R. T. Wets R. J. B. Variational Analysis. Vol. Corrected Edition, Heidelberg Berlin: Springer, 2009.

Russell, R. R.; Schworm, W. "Axiomatic foundations of efficiency measurement on data-generated technologies". Journal of Productivity Analysis 31 (2009):77-86.

———.Technological Inefficiency Indexes: A Binary Taxonomy and a Generic Theorem. UNSW Business School Research Paper No. 2017 ECON 08.

Serra, T., R. G. Chambers, and A. Oude Lansink. "Measuring Technical and Environmental Efficiency in a State-Contingent Technology." European Journal of Operational Research 236 (2014): 706-17.

Shephard, R.W. Cost and Production Functions. Princeton: Princeton University Press, 1953.

——. Theory of Cost and Production Functions. Princeton: Princeton University Press, 1970.

Tone, K."A slacks-based measure of efficiency in data envelopment analysis" European Journal of Operational Research 130 (2001): 498-509.

Zelenyuk, V. and S. Zhao. "Russell and slack-based measures of efficiency: A unifying framework". European Journal of Operational Research (2024, forthcoming).

Zofio, J.L.; Pastor, J.T.; Aparicio, J. "The Directional Profit Efficiency Measure: On Why Profit Inefficiency Is Either Technical or Allocative." Journal of Productivity Analysis 40 (2013): 257-66.

Proofs

Proposition 1

a) Suppose that contrary to the claim that a $z^o \in \partial\delta^*(q|Z)$ for some $q \in ri\ C^*$ and a $z \in Z$ exist for which $z^o - z \in C\backslash\{0\}$. Then

$$\delta^*(z^o - z|C^*) = 0 > q'(z^o - z)$$

because $\delta^*(c|C^*) = \delta(c|C)$ for $c \in C$. But this violates the definition of z^o as belonging to $\partial\delta^*(q|Z)$

b) By definition $z^o \in Eff Z \Rightarrow (z^o - Z) \cap C\backslash\{0\} = \emptyset$ so that $z^o - Z \subset \mathbb{R}^S$ and $C \subset \mathbb{R}^S$ are properly separated. Lemma 2 applies. Choose a $q \in ri\ C^*$ to obtain:

$$\begin{aligned}
\delta^*(q|z^o - Z) &= q'z^o - \delta^*(q|Z) \\
&\geq \inf\{q'x : x \in z^o - Z\} \\
&= 0 \\
&= \delta^*(q|C) \\
&> -\infty
\end{aligned}$$

as required.

Lemma 3

By Proposition 2 and (16):

$$q \in \delta^Q(z|Z) \Leftrightarrow \partial\delta(z - d|Z) \cap \partial\delta^*(d|Q) \Leftrightarrow \delta^Q(z|Z) = \delta(z - d|Z) + \delta^*(d|Q)$$

By (11), $dom\ \delta^*(\cdot|Q) = Q^*_\infty$ and the result follows because $\partial\delta^*(d|Q) = \emptyset$ for $d \notin dom\ \delta^*(\cdot|Q)$.

Proposition 3

a) Immediate from the properties of $\delta^*(q|Z)$ and (12).

b) By Proposition 2

$$z \in \partial\delta^*(q|Z) + d \text{ for } d \in \partial\delta(q|Q).$$

Lemma 3 implies $d \in C$. Thus, (11) and $\delta^Q(z|Z) \leq 0 \Rightarrow q'z \leq \delta^*(q|Z)$ for all q establishing the result.

c) If Q is bounded, then $Q_\infty = \{0\}$, whence $Q^*_\infty = \mathbb{R}^S$. If $\delta^Q(z|Z) > 0$ then $\delta^*(q|z - Z) > 0$ so that $z \notin Z$ by the separating hyperplane theorem.

Proposition 4

We start with a useful lemma. (See, for example, Hiriart-Urruty and LeMarëchal 2001 Proposition E.3.3.1.)

Lemma 4. *Let* $f(x) = \max_{j=1,\ldots,J}\{x'p^j - b_j\}$ *with* $p^j \in \mathbb{R}^{S*}$ *and* $b_j \in \mathbb{R}$ *for all* j*. Then for all* $p \in co\left\{p^1, \ldots, p^J\right\} = dom\ f^*$

$$f^*(p) = \min_\lambda \left\{ \sum_{j=1}^J \lambda_j b_j : \lambda \in \Delta_J \ \ p = \sum_{j=1}^J \lambda_j p^j \right\}$$

where $co\{\cdot\}$ *denotes the convex hull and* $\Delta_J \subset \mathbb{R}^J$ *denotes the unit simplex.*

We now prove the proposition

a) and b) are consequences of the conjugacy between addition and *infimal convolution.* (For example, Rockafellar 1970 Theorem 16.4, Hiriart-Urruty 2001 Theorem E.3.2.1).

c) Let $\delta^Q(z|Z) = \max_{j=1,\ldots,J}\{z'\bar{q}^j - b_j\}$. Then Lemma 4 requires for all $q \in co\left\{\bar{q}^1, \ldots, \bar{q}^J\right\} = dom\ \delta^{Q*}$ that

$$\delta^{Q*}(q|Z) = \min_\lambda \left\{ \sum_{j=1}^J \lambda_j b_j : \lambda \in \Delta_J \ \ q = \sum_{j=1}^J \lambda_j \bar{q}^j \right\}$$

d) Let $z^j \in \mathbb{R}^S,\ j = 1, \ldots, J$ and

$$\delta^*(q|Z) = \begin{cases} \max_{j=1,\ldots,J}\{q'z^j\} & \text{if } q \in C^* \\ \infty & \text{otherwise.} \end{cases}$$

Then using Lemma 4 gives

$$\delta(z^o|Z) = \begin{cases} 0 & \text{if } z^o - \sum_{j=1}^J \lambda_j z^j \in C, \lambda \in \Delta_J \\ \infty & \text{otherwise.} \end{cases}$$

Using (16)

$$\begin{aligned}
\delta^{Q}(z|Z) &= \inf_{z=z^o+d}\{\delta(z^o|Z)+\delta^*(d|Q)\} \\
&= \inf_{z=z^o+d}\left\{\begin{cases} 0 & \text{if } z^o\in\sum_{j=1}^{J}\lambda_j z^j + C, \lambda\in\Delta_J \\ \infty & \text{otherwise.}\end{cases} + \delta^*(d|Q)\right\} \\
&= \inf_{d,\lambda}\left\{\delta^*(d|Q) : z-d\in\sum_{j=1}^{J}\lambda_j z^j + C, \lambda\in\Delta_J\right\}
\end{aligned}$$

for $z\in dom\ \delta^Q$.

e) Let $Z=C$. Then

$$\delta(z|Z)\begin{cases} 0 & \text{if } z\in C \\ \infty & \text{otherwise.}\end{cases}$$

Applying (16) gives the result.

f) Let $Z=\bigcap\limits_k Z^k$ with each $Z^k\subset\mathbb{R}^S,\ k=1,\ \ldots,K$ nonempty closed convex and $\bigcap\limits_k ri\ Z^k\neq\emptyset$. Then $\delta(z|Z)=0\Rightarrow\sum\limits_k\delta\left(z|Z^k\right)=0$. Using the conjugacy between addition and infimal convolution gives

$$\delta^*(q|Z)=\inf_{q^1,\ldots,q^k}\left\{\sum_k\delta^*\left(q^k|Z^k\right):\sum_k q^k=q\right\},$$

and the result then follows from (12).

g) Let $\delta^*(q|Z)=\sum\limits_n\delta^*\left(q|Z^k\right)$. Then the conjugacy between addition and infimal convolution gives

$$\delta(z|Z)=\inf_{z1,\ldots,z^K}\left\{\sum_k\delta\left(z^k|Z^k\right):\sum_k z^k=z\right\},$$

and (16) gives

$$\delta^{Q}(z|Z)=\inf_{z^1,\ldots,z^K}\left\{\sum\delta\left(z^k|Z^k\right)+\delta^*\left(z-\sum_k z^k|Q\right)\right\}.$$

h) Let $Q=Q^1\cap Q^2\cap\cdots\cap Q^K$ with each $Q^k\subset\mathbb{R}^{S*}$ nonempty closed convex and $\bigcap\limits_k ri\ Q^k\neq\emptyset$. Then as in the proof of f)

$$\delta^*\left(d|Q^1\cap Q^2\cap\cdots\cap Q^K\right)=\inf\left\{\sum_k\delta^*\left(d^k|Q^k\right):d=\sum_k d^k\right\},$$

and using (16) gives

$$\begin{aligned}\delta^{Q}\left(z|Z\right) &= \inf_{z=z^{o}+d}\left\{\delta\left(z^{o}|Z\right)+\inf\left\{\sum_{k}\delta^{*}\left(d^{k}|Q^{k}\right):d=\sum_{k}d^{k}\right\}\right\}\\ &= \inf_{d^{1},\ldots,d^{K}}\left\{\delta\left(z-\sum_{k}d^{k}|Z\right)+\sum_{k}\delta^{*}\left(d^{k}|Q^{k}\right)\right\}\end{aligned}$$

.

i) Immediate from the assumption and (16).

j) Let $q^{j}\in\mathbb{R}^{S*},\ j=1,\ldots,J$ and

$$\delta^{*}\left(d|Q\right)=\begin{cases}\max_{j=1,\ldots,J}\left\{d'q^{j}\right\} & \text{if } d\in Q_{\infty}^{*}\\ \infty & \text{otherwise.}\end{cases}$$

Then as in the proof of d)

$$\delta\left(q|Q\right)=\begin{cases}0 & \text{if } q\in\sum_{j=1}^{J}\lambda_{j}q^{j}+Q_{\infty},\lambda\in\Delta_{J}\\ \infty & \text{otherwise.}\end{cases}$$

and the result follows from (12).

k) Immediate.

l) Immediate.

m) Let $0\in Q$. Then, by the properties of gauge functions, $Q=\{q:\gamma\left(q|Q\right)\leq 1\}$. The polarity between gauges and support functions then implies that

$$\begin{aligned}\delta^{*}\left(d|Q\right) &= \sup_{q}\left\{q'd:q\in Q\right\}\\ &= \gamma\left(d|Q_{*}\right)\end{aligned}$$

where $Q_{*}\equiv\left\{d\in\mathbb{R}^{S}:\delta^{*}\left(d|Q\right)\leq 1\right\}$ is closed convex with $0\in Q_{*}$ (for example, Rockafellar 1970 Theorem 14.5). Using (16) gives the result.