Mind the Gap!: Essential Inefficiency Measures

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Abstract

This paper studies the decomposition of economic inefficiency into two components for a class of difference-based technical-inefficiency measures. It develops a formal representation of the economic-inefficiency decomposition into technical-inefficiency and dual-inefficiency components. That dual-inefficiency measure forms part of two dual conjugacies: one with a differenced technical-inefficiency measure and another with a measure of profitability change. The measured is used to investigate an analogue of Aparicio et al.'s (2023) *Essential Property*.

Key words: Inefficiency Measures, Conjugacy, Essential Inefficiency Property

1 Introduction

Since Farrell's (1957) contribution, researchers studying decision-making units (firms) facing price-based markets have decomposed measures of *economic inefficiency* into *technical* and *allocative* components. Economic inefficiency measures how far a firm is from the economically rational outcome for given market prices and technical inefficiency how far it is from the technology's efficient frontier. Allocative inefficiency measures the discrepancy between economic inefficiency and technical inefficiency.

Given market prices, economic efficiency, in either difference or ratio form, measures the distance between two hyperplanes with a common normal defined by the market prices. One hyperplane contains the firm's outcome and the other supports the efficient frontier. Technical inefficiency is identified with the distance between the firm's outcome and a point on the technology's efficient frontier. It is converted into dual (shadow-price) terms using the normal of a hyperplane that supports the efficient frontier at the projected point. For continuous technologies, the efficient frontier is a continuum, and technical inefficiency can be defined as the distance between *any* of these points and the firm's outcome. Because allocative efficiency measures are rooted in differences between economic inefficiency and technical inefficiency, allocative inefficiency measures will vary with technical-inefficiency measures.

When prices do not guide decision making or are not available, inefficiency measurement condenses to measuring technical inefficiency. Following Farrell (1957) and Afriat (1972), a classic approach is to develop a *best-practice frontier* by creating an envelope of an observed data cluster that satisfies key technical axioms. That envelope's boundary defines the best-practice frontier against which to measure performance. The extensions of the Farrell-Afriat framework by Charnes, Cooper, and Rhodes (1978) and Banker, Charnes, and Cooper (1984) established the data envelopment analysis (DEA) model as the canonical framework for measuring technical inefficiency.

Different technical-inefficiency measures have been proposed. Farrell (1957), Charnes et al. (1978), and Banker et al. (1984) emphasized inefficiency measures that projected outcomes onto the frontier by either radial expansion or contraction. Färe and Lovell (1978) demonstrated the equivalence of Farrell's (1957) measure to Shephard's (1953) distance function and suggested a weighted radial expansion as an alternative (the Russell measure). Adapting Luenberger (1992), Chambers, Chung, and Färe (1996, 1998) suggested measuring inefficiency by projecting a firm's outcome onto the frontier along a predetermined (but fixed) direction. Many subsequent contributions offered different technical-inefficiency measures based on different ways to project the firm's outcome to the frontier (for example, Pastor, Sirvent, and Ruiz 1999, Tone 2001, Ray 2005 and 2007, Pastor, Lovell, and Aparicio 2012, Aparicio, Pastor, and Ray 2013, Zofio, Pastor, and Aparicio 2013, Aparicio, Borras, Pastor, Vidal 2015, Petersen 2018).

Studies evaluating competing technical-inefficiency measures often use an *axiomatic approach* that, for the most part, follows a Fisherian "test" approach where technical-inefficiency measures are judged by their ability to satisfy specific axioms (Färe and Lovell 1978, Russell 1985, Pastor et al. 1999, Russell and Schworm 2009 and 2018).[1] Recently, Aparicio, Zofio, and Pastor (2023) have argued that inefficiency measures should also be judged on the basis of "...the properties that the proper decomposition of an economic efficiency index into technical and allocative components should meet". Moreover, they introduce "...properties that are *essential* for the correct interpretation..." of the decomposition (italics added).

This paper investigates the decomposition of economic inefficiency into two components for a class of difference-based technical-inefficiency measures developed for a closed convex feasible set that generalizes the DEA and other representations of technology sets. The focus is on developing a formal representation of the economic-inefficiency decomposition and clarifying the conditions required for it to satisfy an appropriate analogue of Aparicio et al.'s (2023) *Essential Property.* We use pre-specified notions of economic inefficiency and technical-inefficiency to deduce a general measure of allocative inefficiency that shares some characteristics with existing measures. But differences exist, and we call our measure *dual inefficiency* to distinguish it from others. We show that this measure is a proper closed convex bi-function over primal and dual variates that forms part of two dual conjugacies, one with a differenced technical-inefficiency measure and another with a differenced support

[1] Hougaard and Keidig (1998) and Chambers and Miller (2014) are exceptions. They use axioms to deduce specific functional structures.

function. We use the measure to investigate a version of the Aparicio et al. (2023) Essential Property applicable to our framework and state a necessary and sufficient condition for its satisfaction.

The presentation proceeds as follows. We first present basic notation and summarize, for later reference, known results in variational analysis. Next follows our representation of the set of feasible outcomes, the definitions of economic inefficiency and technical inefficiency, and a duality theorem that characterizes the technical-inefficiency measure. We then develop our dual-inefficiency measure using Fenchel's Inequality and the definitions of economic inefficiency and technical inefficiency. We show that the measure is a proper closed convex bi-function and develop necessary and sufficient conditions for it to be zero minimal (Proposition 2). The next section develops two dual conjugacies for the dual inefficiency measure (Propositions 3 and 4). We then discuss the Essential Property of Aparicio et al. (2023). The penultimate section relates our dual-inefficiency measure to other versions of allocative inefficiency and related notions of the Fenchel Inequality. The last section concludes.

2 Notation and Preliminaries

Let $\bar{\mathbb{R}} = [-\infty, \infty]$ and $ri\ X$ denote the relative interior of $X \subset \mathbb{R}^S$. The *effective domain* for a function $f : \mathbb{R}^S \to \bar{\mathbb{R}}$, $dom\ f$, is

$$dom\ f \equiv \left\{ x \in \mathbb{R}^S : f(x) < \infty \right\},$$

and its *subdifferential correspondence*, $\partial f : \mathbb{R}^S \rightrightarrows \mathbb{R}^{S*}$, is[2]

$$\partial f(x) \equiv \left\{ q \in \mathbb{R}^{S*} : q'(z - x) \leq f(z) - f(x),\ \forall z \in \mathbb{R}^S \right\}.$$

For f proper[3] convex, $\partial f(x) \neq \emptyset$ for $x \in ri\,(dom f)$.

The (convex) *conjugate* of f, $f^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, is

$$f^*(q) \equiv \sup_{x \in \mathbb{R}^S} \left\{ q'x - f(x) \right\}. \quad (1)$$

[2] $\mathbb{R}^S$ is, of course, self-dual. We retain the notation, $\mathbb{R}^{S*}$, for its dual space to ensure a clear distinction between dual and primal variates.

[3] A convex function, f, is proper if $dom\ f \neq \emptyset$ and $f(x) > -\infty$ for all x.

f^* is closed convex. For f proper closed[4] convex:

$$\begin{aligned} f^{**}(x) &\equiv \sup_{q \in \mathbb{R}^{S*}} \{q'x - f^*(q)\} \\ &= f(x), \end{aligned} \tag{2}$$

and f^* is also proper. Expressions (1) and (2) form the *conjugacy correspondence*, $f \overset{*}{\longleftrightarrow} f^*$, between proper closed convex f and its proper closed convex conjugate f^*. A well-known consequence is (see, for example, Rockafellar 1970, Moreau 1966, Rockafellar and Wets 2009, Bertsekas 2009)

Lemma 1 *Let f be proper closed convex. Then f^* is proper closed convex,*

$$f(x) + f^*(q) \geq q'x \quad \forall q, x \quad \textit{(Fenchel's Inequality)} \tag{3}$$

$$\hat{q} \in \partial f(\hat{x}) \Leftrightarrow \hat{x} \in \partial f^*(\hat{q}) \Leftrightarrow f(\hat{x}) + f^*(\hat{q}) = \hat{q}'\hat{x}, \tag{4}$$

$$\partial f^*(q) = argmax_x \{q'x - f(x)\} \qquad \partial f(x) = argmax_q \{q'x - f^*(q)\}. \tag{5}$$

Define the *indicator function*, $\delta : \mathbb{R}^S \to \{0, \infty\}$, for $X \subset \mathbb{R}^S$ by

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ \infty & \text{otherwise.} \end{cases} \tag{6}$$

For $X \subset \mathbb{R}^S$ closed convex and nonempty, $\delta(x|X)$ is proper closed and convex. The *support function*, $\delta^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, for X is

$$\begin{aligned} \delta^*(q|X) &\equiv \sup \{q'x : x \in X\} \\ &= \sup_{x \in \mathbb{R}^S} \{q'x - \delta(x|X)\}. \end{aligned} \tag{7}$$

δ^*, as the conjugate of δ, is closed and sublinear. If X is closed nonempty and convex, expression (2) implies δ^* is proper and

$$\delta(x|X) = \sup_{q \in \mathbb{R}^{S*}} \{q'x - \delta^*(q|X)\}. \tag{8}$$

[4] A function is closed if its closure is the function itself. For proper convex functions, closedness is equivalent to lower semi-continuity (Rockafellar 1970, p. 52).

The *gauge function* for $X \subset \mathbb{R}^S$, $\gamma : \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined

$$\gamma\left(x|X\right) = \inf\left\{\gamma > 0 : x \in \gamma X\right\}. \tag{9}$$

If X is nonempty closed convex and $0 \in X$,

$$X = \left\{x : \gamma\left(x|X\right) \leq 1\right\}.$$

The *negative polar cone*[5] of a convex set $X \subset \mathbb{R}^S$ is

$$X^* \equiv \left\{q \in \mathbb{R}^{S*} : q'x \leq 0, \forall x \in X\right\}.$$

For X a closed convex nonempty cone $X^{**} = X$. The *recession (asymptotic) cone* of $X \subset \mathbb{R}^S$ is

$$X_\infty \equiv \left\{d \in \mathbb{R}^S : X + \beta d \subset X, \beta \geq 0\right\}.$$

Here and elsewhere $X + Y$ for $X, Y \subset \mathbb{R}^S$ denotes their *Minkowski sum.* Note that $0 \in X_\infty$ for all $X \subset \mathbb{R}^S$. For X closed convex and nonempty :

$$\begin{aligned} cl\ dom\ \delta^*\left(\cdot|X\right) &= X_\infty^* \\ \left(cl\ dom\ \delta^*\left(\cdot|X\right)\right)^* &= X_\infty \end{aligned} \tag{10}$$

(Hiriart-Urruty and LeMaréchal 2001, Proposition C.2.2.4).

3 Feasible Outcomes and Inefficiency [6]

Let $Z \subset \mathbb{R}^S$ be a closed nonempty convex set that can represent, among others: an input set, an output set, a technology set, or a (convex) envelope of observed data points. We call Z the *technology* and its elements *netputs.*

Define the binary relation, $\preceq_C$, by

$$x \preceq_C y \Leftrightarrow x - y \in C,$$

[5]Some authors refer to X^* as the polar cone of X.

[6]The basic set up and some preliminary results are borrowed from Chambers (2024). To limit unnecessary overlap, we only present those details needed for this paper. Chambers (2024) contains an in-depth discussion of the structure of Z and how it extends more traditional modelling assumptions.

where $C \subset \mathbb{R}^S$ is a closed pointed convex cone. $\preceq_C$ is reflexive, transitive, and antisymmetric so that $(\mathbb{R}^S, \preceq_C)$ forms a *partially ordered set* (Boyd and Vandenberghe 2004, Löhne 2011).

We relate $\preceq_C$ to Z via:

Axiom 1 *$Z_\infty = C$ (where $C \subset \mathbb{R}^S$ defines $\preceq_C$).*

Z_∞ represents the directions in which, starting at a point in Z, one can move towards infinity while staying in the technology. Thus, it corresponds to the directions in which inefficient production remains feasible, or in perhaps more familiar terms netputs are "disposable". Some examples help illustrate different possibilities.

Example 1 *Let $C = \mathbb{R}^S_-$. Then $z \in Z \Rightarrow z' \in Z$ for $z' \leq z$ (free disposability of netputs).*

Example 2 *Let $C = \{0\}$. Then Z is compact.*

Example 3 *Let $C = \{d\}$ where $d \in \mathbb{R}^S$. Then Z satisfies "goodness in the numeraire (d)" in the direction d (Chambers and Färe 2022).*

Because Z is closed convex, its support function, $\delta^*(q|Z)$, is closed sublinear. Thus, Axiom 1 and (10) imply that *dom* $\delta^*(\cdot|Z) = C^*$. So, for example, when $C = \mathbb{R}^S_-$ then *dom* $\delta^*(\cdot|Z) = \mathbb{R}^S_+$, but when $C = \{0\}$ then *dom* $\delta^*(\cdot|Z) = \mathbb{R}^S$. Depending upon $Z's$ interpretation, $\delta^*(q|Z)$ can be, variously, a cost function, a revenue function, a profit function or restricted profit function in *prices* $q \in \mathbb{R}^{S*}$. For concreteness, we call it *profit.*

We call

$$q'z - \delta^*(q|Z) = \delta^*(q|z - Z)$$

the *economic inefficiency for* (z, q). Depending upon the circumstance, it measures excess cost, foregone revenue, or foregone profit incurred by operating at z and is (minus) Nerlove's (1965) efficiency measure.[7] By definition, $\delta^*(q|z - Z) \leq 0$ for all $z \in Z$. The economically rational netput choices, $\partial\delta^*(q|Z)$, are homogenous of degree zero in q by the sublinearity of $\delta^*(q|Z)$.[8]

[7] Chambers (2024) calls this measure Nerlovian inefficiency

[8] Chambers (2024, Proposition 1) relates the economically rational netput choices to an *Efficient Frontier* for Z defined using the Paretian criterion and $\preceq_C$.

When $\delta^* (q|z - Z) < 0$, we call $z \in Z$ $q-$*economically inefficient*. A firm can be economically inefficient at z for some q but not for others. The next definition formalizes the distinction between economic inefficiency and technical inefficiency[9]

Definition 1 *$z \in Z$ is technically inefficient if and only if $\delta^* (q|z - Z) < 0$ for all $q \in ri\ C^*$.*

Technical inefficiency requires that $z \in ri\ Z$ (Hiriart-Urruty and LeMaréchal (2001, Theorem C.2.2.3)) so that the technically-inefficient points fall "inside" Z. To obtain a measure of technical inefficiency, we follow Debreu (1951), Charnes et al. (1978), Luenberger (1992) and seek a $q \in \mathbb{R}^{S*}$ that makes economic inefficiency as close to zero as possible. Because economic inefficiency varies with arbitrary positive scalar multiplication of q, defining an optimization algorithm requires a normalization. Various alternatives have been proposed. They include restricting q to a singleton set, setting the value of a fixed netput bundle to one, restricting q to fall in a closed half space, or restricting q to fall in the intersection of a finite number of closed half spaces of R^S (Debreu 1951, Charnes et al. 1978, Charnes, Cooper, Golany, Seiford, and Stutz 1985, Luenberger 1992, Lovell and Pastor 1995, Ray 2007, Pastor, Lovell, and Aparicio 2012).

We follow Chambers (2024) and require more generally that q to belong to a nonempty closed convex set $Q \subset \mathbb{R}^{S*}$. Our technical-inefficiency measure, $\delta^Q : \mathbb{R}^S \to \bar{\mathbb{R}}$, is defined by maximizing economic inefficiency over Q:

$$
\begin{aligned}
(11) \qquad \delta^Q (z|Z) &\equiv \sup \{q'z - \delta^* (q|Z) : q \in Q\} \\
&= \sup_{q \in \mathbb{R}^{S*}} \{q'z - \delta^* (q|Z) - \delta (q|Q)\}.
\end{aligned}
$$

$\delta^Q (z|Z)$, in effect, chooses q to evaluate z's performance in the most favorable terms possible relative to Q. Note its structural similarity to (8). The next proposition is established in Chambers (2024).

Proposition 1 *$\delta^Q : \mathbb{R}^S \to \bar{\mathbb{R}}$ is proper closed convex. Moreover,*

$$
(12) \qquad \delta^Q (z|Z) \overset{*}{\longleftrightarrow} \delta^* (q|Z) + \delta (q|Q)
$$

[9]Chambers (2024) calls this condition Paretian Inefficiency.

$$\delta^{Q}(z|Z) = \inf_{d \in \mathbb{R}^S} \{\delta^*(d|Q) : z - d \in Z\} \tag{13}$$

$$\delta^{Q}(z|Z) + \delta^*(q|Z) + \delta(q|Q) \geq q'z \ \ \forall q, z \tag{14}$$

$$\hat{q} \in \partial\delta^{Q}(\hat{z}|Z) \Leftrightarrow \hat{z} \in \partial\delta^*(\hat{q}|Z) + \partial\delta(\hat{q}|Q) \Leftrightarrow \delta^{Q}(\hat{z}|Z) + \delta^*(\hat{q}|Z) + \delta(\hat{q}|Q) = \hat{q}'\hat{z}, \tag{15}$$

$$\hat{q} \in \partial\delta^{Q}(\hat{z}|Z) \Leftrightarrow \hat{q} \in \partial\delta^*\left(\hat{d}|Q\right) \cap \partial\delta\left(\hat{z} - \hat{d}|Z\right) \Leftrightarrow \delta^{Q}(\hat{z}|Z) = \delta^*\left(\hat{d}|Q\right) + \delta\left(\hat{z} - \hat{d}|Z\right) \tag{16}$$

Expression (13) follows because

$$\begin{aligned} \delta^{Q}(z|Z) &= \sup_{q \in \mathbb{R}^{S*}} \left\{ q'z - \delta^*(q|Z) - \delta(q|Q) \right\} \\ &= \sup_{q \in \mathbb{R}^{S*}} \left\{ q'(z^o + d) - \delta^*(q|Z) - \delta(q|Q) : z = z^o + d \right\} \\ &= \inf_{(z^o, d)} \left\{ \delta(z^o|Z) + \delta^*(d|Q) : z = z^o + d \right\} \end{aligned} \tag{17}$$

and shows that the maximum difference between the hyperplanes through z and supporting Z with q restricted to fall in Q (our technical-inefficiency measure) is given by the minimal support for Q of the difference between z and an element of $Eff\ Z$. It has a direct parallel in Nirnberg's Minimum-Norm Theorem. The other components of Proposition 1 follow by the proper closed convexity of $\delta^*(q|Z) + \delta(q|Q)$ and Lemma 1. Expression (15) generalizes the familiar envelope theorem and requires a link between the outcome, z, and $\partial\delta^*(q|Z)$ via the subdifferential of $\delta(q|Q)$. Following Ray (2007), we call elements of $\partial\delta^*(q|Z)$ for the minimizing q *endogenous projections* and the linking $\partial\delta(q|Q)$ *endogenous directions.* Chambers (2024, Proposition 4) presents a series of composition results for special cases of Z and Q based on Proposition 1 that covers a range of different measures that are special cases of δ^Q

4 Fenchel's Inequality and Dual Inefficiency

Using (14) gives

$$\delta^{Q}(z|Z) + \delta(q|Q) \geq \delta^*(q|z - Z) \tag{18}$$

for all (z, q) as a proper closed convex upper bound to economic inefficiency. Define $\varphi^Q : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}_+$ by

$$\begin{aligned} \varphi^Q(q, z|Z) &\equiv \delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q) - q'z \\ &= \begin{cases} \delta^Q(z|Q) + \delta^*(q|Z) - q'z & \text{if } q \in Q \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

φ^Q is proper closed convex as a function of z and proper closed convex as a function of q, has *dom* $\delta^Q \subset Q \times R^S$, closes any duality gap implied by a strict inequality in (18), and *for* $q \in Q$ measures the difference between $z's$ technical inefficiency and its economic inefficiency. Thus,

$$\delta^*(q|z - Z) = \delta^Q(z|Z) - \varphi^Q(q, z|Z), \tag{19}$$

for $q \in Q$ and all z.

Expression (19) evokes the interpretation of $\varphi^Q(q, z|Z)$ as the *allocative component* of economic inefficiency. Interpretive issues intrude, however, for $q \notin Q$. Then $\varphi^Q(q, z|Z) = \infty$ suggesting that the firm made an arbitrarily large allocative error when, in fact, economic inefficiency and technical inefficiency are not commensurable for $(z, q) \notin$ *dom* φ^Q. While $\varphi^Q(q, z|Z)$ closes a residual gap between economic inefficiency and technical inefficiency over Q, more generally, it closes any "duality gap" between $\delta^q(z|Z)$ and $\delta^*(q|z - Z) + \delta(q|Q)$. Here $\delta(q|Q)$ operationalizes the restriction of economic inefficiency, and thus $\delta^Q(z|Z)$, to Q. Thus, a direct comparison between $\delta^Q(z|Z)$ and $\delta^*(q|Z)$ is only proper when $q \in Q$. To emphasize this distinction, we call $\varphi^Q(q, z|Z)$ the *dual inefficiency at* (z, q).

By definition, $\varphi^Q(q, z|Z) \geq 0$ for all (z, q). An immediate consequence of its definition and Proposition 1 is

Proposition 2 $\varphi^Q(q, z|Z)$ *is zero minimal if and only if*

$$q \in \partial\delta^Q(z|Z) \Leftrightarrow z \in \partial\delta^*(q|Z) + \partial\delta(q|Q).$$

4.1 Conjugacy Correspondences for φ^Q

The dual inefficiency measure, $\varphi^Q(q,z|Z)$, is closed convex in q and closed convex in z. When viewed from a Lagrangean perspective

$$\varphi^Q(q,z|Z) = \delta^Q(z|Z) + \delta^*(q|Z) + \delta(q|Q) - q'z$$

admits two parallel interpretations. One, as the Lagrangean function for a closed convex program for minimizing $\delta^Q(z|Z)+\delta^*(q|Z)+\delta(q|Q)$ over q. And another, as the Lagrangean function for a closed convex program for minimizing $\delta^Q(z|Z)+\delta^*(q|Z)+\delta(q|Q)$ over z. In the first, z plays the role of a Lagrange multiplier and q plays that role in the second.

That observation implies that φ^Q has "a life of its own" as part of at least two well-defined conjugacy correspondences. We now show that φ^Q is conjugate dual to a measure of the change in technical inefficiency and conjugate dual to a measure of (restricted) profit change.

Define $\varphi^{Q**} : \mathbb{R}^S \times \mathbb{R}^S \to \bar{\mathbb{R}}$

$$\begin{aligned}\varphi^{Q**}(\bar{z},z|Z) &\equiv \sup_{q\in\mathbb{R}^{S*}} \left\{ q'\bar{z} - \varphi^Q(q,z|Z) \right\} \\ &= \delta^Q(\bar{z}+z|Z) - \delta^Q(z|Z).\end{aligned} \tag{20}$$

Expression (20) defines the (partial) conjugate of $\varphi^Q(q,z|Z)$ treated as a proper closed convex function of q. The second equality follows by (11) and Proposition 1 and shows that $\varphi^{Q**}(\bar{z},z|Z)$ is the change in technical inefficiency incurred in moving from z to $\bar{z}$. It is proper closed convex in $\bar{z}$. Applying Lemma 2 gives (note that the dual conjugacy here, $\overset{*}{\longleftrightarrow}$, is between $(\bar{z},q)$ for given z):

Proposition 3 $\varphi^Q(q,z|Z) \overset{*}{\longleftrightarrow} \delta^Q(\bar{z}+z|Z) - \delta^Q(z|Z)$

$$\varphi^Q(q,z|Z) + \delta^Q(\bar{z}+z|Z) - \delta^Q(z|Z) \geq q'\bar{z} \quad \forall q,\bar{z} \quad \textit{(Fenchel's Inequality)}$$

$$\hat{z} \in \partial_q \varphi^Q(q,z|Z) \Leftrightarrow q \in \partial_{\hat{z}} \delta^Q(\hat{z}+z|Z) \Leftrightarrow \varphi^Q(q,z|Z) + \delta^Q(\bar{z}+z|Z) - \delta^Q(z|Z) = q'\hat{z}$$

Symmetrically, define $\varphi^{Q*} : \mathbb{R}^{S*} \times \mathbb{R}^{S*} \to \bar{\mathbb{R}}$ as the (partial) conjugate of φ^Q treated as a closed convex function of z:

$$\begin{aligned}\varphi^{Q*}(q,\bar{q}|Z) &\equiv \sup_{z\in\mathbb{R}^S} \left\{ \bar{q}'z - \varphi^Q(q,z|Z) \right\} \\ &= \delta^*(\bar{q}+q|Z) + \delta(\bar{q}+q|Q) - \delta^*(q|Z) - \delta(q|Q),\end{aligned} \tag{21}$$

where the second equality follows by (11) and Proposition 1 and establishes that $\varphi^{Q*}(q,\bar{q}|Z)$ is the change in (restricted) profit associated with moving from q to $\bar{q}$. Hence,

Proposition 4 $\varphi^{Q}(q,z|Z) \overset{*}{\longleftrightarrow} \delta^{*}(\bar{q}+q|Z)+\delta(\bar{q}+q|Q)-\delta^{*}(q|Z)-\delta(q|Q)$

$$\varphi^{Q}(q,z|Z)+\delta^{*}(\bar{q}+q|Z)+\delta(\bar{q}+q|Q)-\delta^{*}(q|Z)-\delta(q|Q) \geq \bar{q}'z \ \ \forall \bar{q},z \quad \textit{(Fenchel's Inequality)}$$

$$\begin{array}{rcl} \hat{q} \in \partial_{z}\varphi^{Q}(q,z|Z) & \Leftrightarrow & z \in \partial_{\hat{q}}\delta^{*}(\hat{q}+q|Z)+\partial_{\hat{q}}\delta(\hat{q}+q|Q) \\ & \Updownarrow & \\ \hat{q}'z-\delta^{*}(\hat{q}+q|Z)-\delta(\hat{q}+q|Q) & = & \varphi^{Q}(q,z|Z)-\delta^{*}(q|Z)-\delta(q|Q) \end{array}$$

The dual-inefficiency measure is conjugate dual to a) a measure of Paretian inefficiency change and b) a measure of relative profitability. Propositions 3 and 4 are direct consequences of Lemma 1 and Proposition 1. Indeed, they convey the same mathematical information. Their analytic force lies in the recognition that $\varphi^{Q}(q,z|Z)$, despite its derivation as a residual, forms a component of a two conjugacy correspondences for two proper closed convex inefficiency measures. So, for example, specification of a $\varphi^{Q}(q,z|Z)$ that is proper closed convex in q and proper closed convex in z implies the existence of a well-behaved $\delta^{Q}(\bar{z}+q|Z)-\delta^{Q}(z|Z)$ and *vice versa* without the need for "...difficult constructive arguments" (McFadden 1978). Thus, once a proper closed convex $\delta^{Q}(z|Z)$ is chosen, Proposition 3 provides an algorithim for generating a proper closed convex $\varphi^{Q}(q,z|Z)$ without the need for a direct specification of Q and $\delta^{*}(q|Z)$. Conversely, specification of a closed convex $\varphi^{Q}(q,z|Z)$ generates a closed convex function of $\bar{z}$ that measures the change in technical efficiency associated with moving from z to $\bar{z}$. Parallel logic applies to $\delta^{*}(q|Z)$ and $\varphi^{Q}(q,z|Z)$.

4.2 Essential Inefficiency Measures

Until recently, axiomatic analyses of inefficiency measures focused attention on several key properties that a measure should possess without focusing attention on the decomposition of economic inefficiency. Aparicio et al. (2023, p. 115) have argued that evaluations of technical-inefficiency measures should also direct attention to "...the properties that the decomposition of the economic efficiency index into technical and allocative components should meet". Moreover, they suggest that an allocative inefficiency measure, $A : \mathbb{R}^{S*} \times \mathbb{R}^S \to \bar{\mathbb{R}}$, should satisfy an *Essential Property* that requires allocative inefficiency to equal zero in the case of difference-based efficiency measurement and one for ratio-based inefficiency measurement when z is economically efficient for q.[10]

This paper focuses on the implications of Aparicio et al.'s (2023) Essential Property and leaves the discussion of other axioms to other fora. An analogue of that property for our difference based setting is:

Property 1 *If $z \in \partial\delta^* (q|Z)$, then $A(q, z) = 0$.*

Aparicio et al. (2023) show that the allocative counterparts of the Russell Graph Measure, the Enhanced Russell Graph, the Additive, and the Weighted Additive measures do not satisfy Property 1. Ray (2005) shows that the Russell measure is a scale-invariant version of the Additive and Weighted Additive measures. Therefore, following Ray (2005), an analogous class of technical-inefficiency measures is defined by those measures for which $Q = \{\bar{q}\}$ (a singleton set). Then $\delta^Q (z|Z) = \delta^* (\bar{q}|z - Z)$, and

$$\varphi^Q (q, z|Z) = \begin{cases} 0 & \text{if } q = \bar{q} \\ \infty & \text{otherwise.} \end{cases} \tag{22}$$

Dual inefficiency derived for this class of technical inefficiency measures only satisfies Property 1 when $q = \bar{q}$. Elsewhere, it fails. More generally, we have as a direct consequence of Proposition 2 that the requirement for $\varphi^Q (z, q|Q)$ is

Proposition 5 *Let $z \in \partial\delta^* (q|Z)$. Then $\varphi^Q (q, z|Z) = 0$ if and only if*

$$\partial\delta (q|Q) \cap (z - \partial\delta^* (q|Z)) \neq \emptyset$$

[10] Aparicio et al. (2023) provide an extensive discussion of a variety of inefficiency measures and commonly maintained axioms. Their discussion segregates inputs and outputs and discusses separate input and output-oriented versions of their Essential Property for both ratio-based and difference-based measures. It states their Essential Property for the varying cases in Definitions 1-6. It also examines an extended Essential Property and treats the case of singleton $\partial\delta^* (q|Z)$ separately from the more general case. Rather than devote space to parallel restatements of results, we only treat the Essential Property for difference-based measures for a netput vector. Relevant extensions are left to the reader.

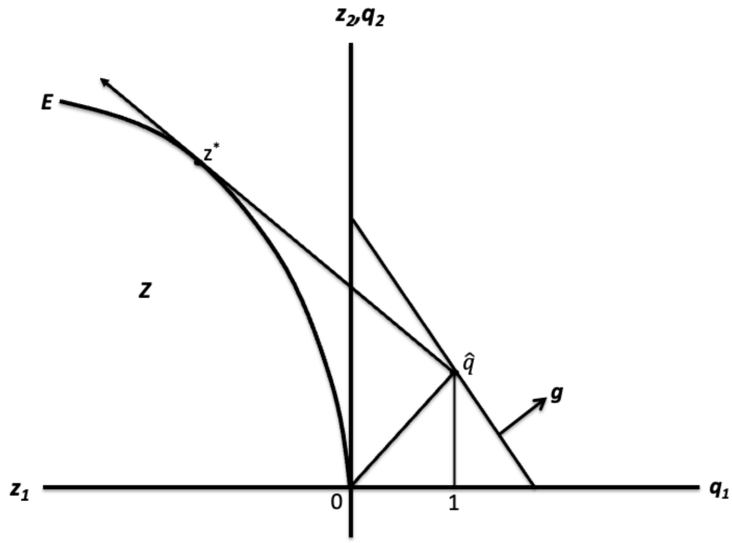

Figure 1: Failure of Essential Property

The set of potential endogenous directions for q must overlap the Minkowski difference between the firm's outcome, z, and the set of potential endogenous projections for q. Figure 1 illustrates the types of issues that might occur. Let $Z_\infty = \mathbb{R}^2_-$ and the efficient set be the curve labelled $0E$. First, let Q be the closed line segment connecting $(1,0)$ and $\hat{q}$. Then the set of efficient projections consistent with Q is restricted to points falling below z^* on OE. In particular, points on the arc beyond z^* on OE cannot be efficient projections. Thus, if q supports one of those efficient projections then $\partial\delta\left(q|Q\right) = \emptyset$. The class of inefficiency measures generated by taking Q to be a singleton set is a polar version of this problem for which $\partial\delta(q|Q) = \emptyset$ almost everywhere.

Now, let $Q = \{q : q'g = 1\}$ (a directional distance function) for the g illustrated there. Q now encompasses all directions in C^* so that no point on OE is prohibited from being an endogenous projection. φ^Q derived from these measures satisfies Property 1.[11]

The fundamental difference between these two illustrative normalization schemes is that

[11] Aparicio et al. (2023) show that the decomposition associated with the directional-distance functions satisfies their Essential Property.

taking Q to equal the closed line segment restricts nominal prices and limits the range of real prices. Thus, it arbitrarily excludes whole regions of $Eff\ Z$ from serving as endogenous projections. The directional-distance function also restricts nominal prices, but it allows real prices to vary enough to support $Eff\ Z$. Some inefficiency normalizations do more than normalize nominal prices. They also rule out potentially relevant real prices from supporting endogenous projections. The fixed (dual) weight technical inefficiency measures such as the additive, the weighted additive, and the Russell measures exemplify the problem for allocative measures derived by differencing economic and technical inefficiency.

4.3 Fenchel's Inequality and Generalized or Extended Fenchel Inequalities

Our use of Fenchel Inequality's, (18), differs from some related invocations of that principle. Some specific examples help illustrate. First, let $Q = \left\{q \in \mathbb{R}^{S*} : q'g = 1\right\}$ with $g \in Z_\infty$. Then

$$\text{(23)} \qquad \begin{aligned} \delta^Q(z|Z) &= \sup_q \{q'z - \delta^*(q|Z) : q'g = 1\} \\ &= \sup_q \left\{ \frac{q}{q'g}' z - \delta^* \left(\frac{q}{q'g} | Z \right) \right\} \end{aligned}$$

is the manifestation of Luenberger's (1992) shortage function that corresponds to the directional distance function of Chambers et al. (1996, 1998). Reasoning parallel to (3), Chambers et al. (1998) noted that:

$$\text{(24)} \qquad \begin{aligned} \delta^Q(z|Z) &\geq \frac{q}{q'g}' z - \delta^* \left(\frac{q}{q'g} | Z \right) \\ &= \delta^* \left(\frac{q}{q'g} | z - Z \right) \text{ for all } q, z. \end{aligned}$$

The technical-inefficiency measure for z bounds economic inefficiency evaluated at $\frac{q}{q'g}$ from above. Using this observation, Chambers et al. (1998) then defined (minus) allocative efficiency in residual terms as the difference between $\frac{q}{q'g}-$ economic inefficiency and technical inefficiency.[12] Note, as illustrated above, that the decomposition associated with the

[12] Chambers et al. (1998) and many following authors define economic and technical inefficiency as minus one times our corresponding measures. Hence, the sign difference.

directional-distance function satisfies Property 1.

As another example, let $Q = \left\{ q \in \mathbb{R}^{S*} : \sum_{s \in 1,\ldots,S} |q_s| \leq 1 \right\}$ (the unit disc for the taxicab norm). Then

$$\begin{aligned} \delta^Q(z|Z) &= \sup_q \left\{ q'z - \delta^*(q|Z) : \sum_{s\in 1,\ldots,S} |q_s| \leq 1 \right\} \\ &= \sup_q \left\{ \frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}' z - \delta^* \left(\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|} |Z \right) \right\} \end{aligned} \tag{25}$$

Hence,

$$\delta^Q(z|Z) \geq \frac{q}{\sum_{s\in 1,\ldots,S} |q_s|}' z - \delta^* \left(\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|} |Z \right) \text{ for all } q \neq 0, z, \tag{26}$$

which evokes the interpretation of

$$\delta^Q(z|Z) - \delta^* \left(\frac{q}{\sum_{s\in 1,\ldots,S} |q_s|} |Z - z \right)$$

as the allocative component of normalized economic inefficiency. Using (13) gives

$$\begin{aligned} \delta^Q(z|Z) &= \inf_{z = z^o + d} \left\{ \delta(z^o|Z) + \delta^*(d|Q) \right\} \\ &= \inf_d \left\{ \max_{s\in 1,\ldots,S} \{|d_s|\} : z - d \in Z \right\} \end{aligned} \tag{27}$$

Without loss of generality, let z_1 be the minimizer for (27). For real units to be comparable, the right-hand side of (26) now must be evaluated in units of z_1. Because the element of z that optimizes (25) can vary with z, the numeraire will vary with z.

Different writers name (24), (26), and their difference-based or ratio-based analogues differently, including the *Luenberger Inequality*, the *Fenchel-Mähler Inequality*, the *generalized Fenchel-Young Inequality*, among others (see, for example, Färe and Grosskopf 1997, Chambers and Färe 2004, Cooper et al. 2011, Zofio et al. 2013, Aparicio et al. 2015, Petersen 2018).[13] All convey the same message. A gap between a *real* measure of economic ineffi-

[13] For $X \subset \mathbb{R}^S$ closed convex and containing the origin, its gauge and support functions are polar to one another. Hence,

$$\delta^*(q|X) = \sup \{q'x : \gamma(x|X) \leq 1\},$$

whence $\delta^*(q|X)\gamma(x|X) \geq 1$, which manifests the *Mähler-Gorman Inequality.*

ciency and technical inefficiency is closed by subtraction, and the resulting residual is called *allocative inefficiency* (see Aparicio et al. (2023) for an extended discussion).

Expressions (18), (24), and (26) manifest the same phenomenon but use different mathematical formulations to describe it. Expression (18) gives a dual inefficiency measure that closes a "duality gap" by subtraction for all (z, q). It operates in dual units restricted to lie in Q so that when $q \in Q$ it closes the gap between economic inefficiency and technical efficiency. Expression (24), a residual allocative-inefficiency measure obtained by subtraction, is expressed in real units of the direction g. It follows from (18) and after normalization of prices closes the gap between economic inefficiency and technical inefficiency. It predetermines the endogenous direction to be g. Expression (26) also yields a real dual inefficiency measure by subtraction. It, too, follows from (18), but unlike expression (24) it does not predetermine the endogenous direction.

5 Concluding Remarks

We investigate the decomposition of economic inefficiency into two components for a class of difference-based technical-inefficiency measures for a broad class of convex feasible sets. We develop a formal representation of the economic-inefficiency decomposition into technical-inefficiency and dual-inefficiency components. We show that the dual-inefficiency measure forms part of two dual conjugacies: one with a differenced technical-inefficiency measure and another with a measure of profitability change. We then use the measure to investigate an analogue of Aparicio et al.'s (2023) *Essential Property* and to develop necessary and sufficient conditions for the measure to satisfy that analogous property.

Proposition 3 establishes that starting with a proper closed convex $\delta^{Q}(z|Z)$, one can avoid the difficulty of constructing an inefficiency-measurement system for technical and dual efficiency from scratch by specifying Z and Q and then solving (11). Proposition 4 establishes that one can start with $\delta^{*}(q|Z)$ and Q to obtain a measure of dual inefficiency directly without explicitly solving for $\delta^{Q}(z|Z)$.

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7 Data Availability

: No data are used in this paper.