Overall Inefficiency Measures: A Different Perspective

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Abstract

We study the Ray (2007) overall inefficiency measure and its generalization by Aparicio et al. (2013) for general convex technologies. We show that it admits an alternative interpretation as measuring the maximal input (output) inefficiency consistent with projecting an observed input-output bundle by the *same proportion* in predetermined directions. We relate this interpretation to elasticity measures defined by Balk et al. (2015) and Podinovski et al. (2016) and show that the overall inefficiency measure reduces to a directional input (output) inefficiency measure for translation-homothetic technologies. A dual formulation of the overall inefficiency measure shows that it corresponds to the saddlepoint of a problem that computes a " price-restricted indicator function".

Key words: data envelopment analysis, overall inefficiency measure, translation homotheticity, input-output inefficiency, endogenous shadow prices

Introduction

Ray (2007) defined an *overall inefficiency* measure as the solution to a *shadow profit maximization* problem. Aparicio, Pastor, and Ray (2013) developed a generalization of Ray's (2007) measure that also generalized Luenberger's (1992) shortage function. A defining characteristic of the Ray (2007) and Aparicio et al. (2013) measures is that they allow input and output adjustment by *different proportions* in predetermined input-output directions $(-v, u)$.

We study such measures for general convex technologies that incorporate the canonical DEA framework as an important special case. We show that the resulting overall inefficiency measure admits an alternative interpretation as measuring the maximal input (output) inefficiency consistent with projecting an observed input-output bundle by the *same proportion* in the direction (v, u). We use that observation to relate the overall inefficiency measure to elasticity measures defined by Balk, Färe, and Karagiannis (2015) and Podinovski, Chambers, Atici, and Deineko (2016) and to show that the overall inefficiency measure reduces to a directional input (output) inefficiency measure for translation-homothetic technologies. The innate connection between the overall inefficiency measure and translation-homothetic technologies ensures that it, unlike the directional measures it generalizes, does not provide an exhaustive cardinal characterization of T. We use a dual-space formulation of the overall inefficiency measure to reveal its natural interpretation as a "price-restricted indicator function".

In what follows, we first introduce basic notation, the model of the technology, and standard results on directional inefficiency measurement. We then demonstrate our alternative interpretation, its relation to measures of translatability and translation-homothetic technologies, and characterize the measure for convex technologies. The penultimate section presents a dual-space formulation of the overall inefficiency measure, and the final section concludes.

Notation and Model

Let $\bar{\mathbb{R}} \equiv [-\infty, \infty]$ denote the extended real numbers. Define the *subdifferential* correspondence, $\partial_x f : \mathbb{R}^K \rightrightarrows \mathbb{R}^{K*}$, for $f : \mathbb{R}^K \to \bar{\mathbb{R}}$ proper and closed convex by[1]

$$\partial_x f(x) \equiv \left\{ q \in \mathbb{R}^{K*} : q'(y-x) \le f(y) - f(x) \quad \forall y \in \mathbb{R}^K \right\}$$

and $f's$ (one-sided) directional derivative (in the direction v) by

$$D_x^v \circ f(x) \quad \equiv \quad lim_{\lambda \downarrow 0} \frac{f(x+\lambda v) - f(x)}{\lambda}.$$

Let the *technology* be represented by $T \subset \mathbb{R}_+^{N+M}$ that is nonempty, closed, and convex. For given $v \in \mathbb{R}_+^N \setminus \{0\}, u \in \mathbb{R}_+^M \setminus \{0\}$, we assume that T satisfies

$$(x, y) \in T \Rightarrow (x + \mu v, y - \lambda u) \in T \qquad \mu \ge 0, \lambda \ge 0, \ (x + \mu v, y - \lambda u) \in \mathbb{R}_+^{N+M}. \tag{1}$$

We refer to (1) as *goodness in the direction* $(v, -u)$ and note that it weakens *free disposability of inputs and outputs* to monotonicity the direction $(v, -u)$. Define the *directional input inefficiency measure*, $I_x^v : \mathbb{R}_+^{N+M} \to \bar{\mathbb{R}}$, as

$$I_x^v(x, y) \equiv \inf \{ \gamma \in \mathbb{R} : (x + \gamma v, y) \in T \} \tag{2}$$

and the *directional output inefficiency measure*, $I_y^u : \mathbb{R}_+^{M+N} \to \bar{\mathbb{R}}$, as

$$I_y^u(y, x) \equiv \inf \{ \mu \in \mathbb{R} : (x, y - \mu u) \in T \}. \tag{3}$$

The input and output inefficiency measures are versions of Luenberger's (1992) shortage function and correspond, respectively, to (minus) input and output directional distance functions.

We have (all proofs are in an Appendix):

Lemma 1 *$I_x^v(x, y)$ satisfies: a) $I_x^v(x, y) \le 0 \Leftrightarrow (x, y) \in T$ (Indication); b) $I_x^v(x + \alpha v, y) = I_x^v(x, y) - \alpha, \quad \alpha \in \mathbb{R}$ (Translation); c) $I_x^v(x, y)$ is nonincreasing in the direction $(v, -u)$ (Monotonicity); and d) convex in (x, y). (Convexity)*

Lemma 2 *$I_y^u(y, x)$ satisfies: a) $I_y^u(x, y) \le 0 \Leftrightarrow (x, y) \in T$ (Indication); b) $I_y^u(y + \alpha u, x) = I_y^u(y, x) + \alpha, \quad \alpha \in \mathbb{R}$ (Translation); c) $I_x^u(y, x)$ is nonincreasing in the direction $(v, -u)$; and d) convex in (x, y) (Convexity).*

[1] $\mathbb{R}^{K*}$ is the dual space of $\mathbb{R}^K$ and equals $\mathbb{R}^K$. We preserve the notational distinction to clarify which maps are defined in terms of primal variables and which are defined in terms of dual variables.

Alternative Representations of Overall Inefficiency

Following Ray (2007) and Aparicio et al. (2013), define the *generalized overall inefficiency measure*, $O : \mathbb{R}_{+}^{N} \times \mathbb{R}_{+}^{M} \to \bar{\mathbb{R}}$, as:

$$O_v^u(x, y) \equiv \sup \{\beta + \phi : (x - \beta v, y + \phi u) \in T\}. \tag{4}$$

We follow Ray (2007) and leave $\beta, \phi \in \mathbb{R}$ free. Aparicio et al. (2013) restrict both to be positive, so formulation (4) is a slight generalization of their measure.[2] Our first results provide alternative representations of $O_v^u(x, y)$.

Proposition 3

$$\begin{aligned} O_v^u(x, y) &= \sup_{\phi \in \mathbb{R}} \{\phi - I_x^v(x, y + \phi u)\} && (5) \\ &= -\inf_{\phi \in \mathbb{R}} \{I_x^v(x + \phi v, y + \phi u)\} && (6) \end{aligned}$$

Proposition 4

$$\begin{aligned} O_x^u(x, y) &= sup_{\beta \in \mathbb{R}} \left\{\beta - I_y^u(y, x - \beta v)\right\} && (7) \\ &= -inf_{\beta \in \mathbb{R}} \left\{I_y^u(y - \beta u, x - \beta v)\right\} && (8) \end{aligned}$$

Propositions 3 and 4 offer different perspectives on the overall inefficiency problem. Where Ray (2007) and Aparicio et al. (2013) emphasize its interpretation as a generalized directional distance measure, these propositions show that $O_v^u(x, y)$ also measures inefficiency associated with translating (x, y) by the *same proportion* in the direction (v, u) instead of $(-v, u)$.[3]

Figure 1 illustrates version (5) in Proposition 3 for an $(x, y) \in T$.[4] In their study of marginal valuation for polyhedral technologies, Podinovski et al. (2016) call $I_x^v(x, y + \phi u)$ a *directional response function.* It measures the maximal amount that x can be translated in the direction v, while keeping $y + \phi u$ technically feasible. By Lemma 1, it is nonpositive at

[2] Aparicio et al. (2013) work in a DEA framework . Thus, the inequality restriction is more natural in their setting than in ours.

[3] Ray and Yang (2022, p.22, their expression (25)) report an analogue of Proposition 4 in a single-input, single-output setting.

[4] Parallel visual illustrations exist for Proposition 4 but are not treated to conserve space.

$\phi = 0$ and nondecreasing and convex as a function of ϕ. Proposition 3 shows that $O_x^u(x, y)$ maximizes the difference between the proportional movement of y in the direction of u and this directional response function. That difference is illustrated in Figure 1 by the vertical distance between the line through the origin with slope of 1 (the bisector) and the graph of $I_x^v(x, y + \phi u)$. The maximal difference occurs where the slope of the graph $I_x^v(x, y + \phi u)$ in ϕ equals the slope of the line. But when $I_v^u(x, y + \phi u)$ is smooth, that slope equals

$$\lim_{\phi - \phi^* \to 0} \frac{I_x^v(x+, y + (\phi - \phi^*) u + \phi^* u) - I_x^v(x, y + \phi u)}{\phi - \phi^*} = D_y^u \circ I(x, y + \phi^* u).$$

Figure 2 illustrates formulation (6). The task is now to find the projection of (x, y) in the direction (v, u) that achieves maximal directional input inefficiency. That maximal difference occurs where the horizontal distance (more generally, in the direction v) between the projections of (x, y) in the direction (v, u)

$$\{(\hat{x}, \hat{y}) : (\hat{x}, \hat{y}) = (x + \phi v, y + \phi u), \quad \phi \in \mathbb{R}\},$$

and $T's$ boundary is the largest. That requires that the slope of $T's$ boundary equal that of the line segment describing the projections. The point to which (x, y) is projected on the hyperplane is $(x + \phi^* v, y + \phi^* u)$ while $(x - \beta^* v, y + \phi^* u)$ depicts the corresponding solution to formulation (4).

We formalize these visual arguments in:

Proposition 5

$$\begin{aligned} O_v^u(x, y) &= \phi^* - I_x^v(x, y + \phi^* u) \\ &= -I_x^v(x + \phi^* v, y + \phi^* u) \end{aligned}$$

if and only if

$$D_y^u \circ I_x^v(x, y + \phi^* u) \geq 1 \geq -D_y^{-u} \circ I_x^v(x, y + \phi^* u) \tag{9}$$

and

$$D_{(x,y)}^{(v,u)} \circ I_x^v(x + \phi^* v, y + \phi^* u) \geq 0 \geq -D_{(x,y)}^{-(v,u)} \circ I_x^v(x + \phi^* v, y + \phi^* u) \tag{10}$$

Proposition 6

$$\begin{aligned} O_v^u(x,y) &= \beta^* - I_x^y(y, x - \beta^* u) \\ &= -I_x^y(y - \beta^* u, x - \beta^* u) \end{aligned}$$

if and only if

$$D_x^v \circ I_y^u(y, x - \beta^* v) \geq 1 \geq -D_x^{-v} \circ I_y^u(y, x - \beta^* v)$$

and

$$D_{(y,x)}^{(u,v)} \circ I_y^u(y - \beta^* u, x - \beta^* v) \geq 0 \geq -D_{(y,x)}^{-(u,v)} \circ I_y^u(y - \beta^* u, x - \beta^* v)$$

Propositions 5 and 6 show that a) $(x + \phi^* v, y + \phi^* u)$ and b) $(x - \beta^* v, y - \beta^* u)$ occur, respectively, where:

a) A one unit proportional movement of y in the direction of u requires at least a one unit proportional movement of x in the direction of v to keep inefficiency constant, and a unit proportional movement of y in the direction of $-u$ requires no more than a one unit proportional movement of x in the direction of $-v$ to keep inefficiency constant.

b) A one unit proportional movement of x in the direction of v requires at least a one unit proportional movement of y in the direction of u to keep inefficiency constant, and a unit proportional movement of x in the direction of $-v$ requires no more than a one unit proportional movement of y in the direction of $-u$ to keep inefficiency constant.

By analogy with Balk, Färe, and Karagiannis (2015) and Podinovski et al. (2016), we refer to Propositions 5 and 6 as requiring *constant returns to translation* in the direction (v, u) at $(x + \phi^* v, y + \phi^* u)$ and $(x - \beta^* v, y - \beta^* u)$.[5]

These propositions suggest that technologies that are *translatable* in the direction (v, u) possess uniquely tractable versions of O_v^u. Our next result verifies this intuition. Following

[5] Balk et al. (2015) develop results for everywhere smooth (differentiable) T. The convexity of $I_x^v(x, y + \phi u)$ in ϕ ensures that it is almost everywhere differentiable when it is finite. Thus, their analysis applies almost everywhere. We develop their measure in our notation in an Appendix to verify its equivalence to the forms in Propositions 5 and 6. Nevertheless, O_v^u was formulated in a DEA setting where smoothness violations at efficient points are routine. Propositions 5 and 6 cover these cases. Linear programming algorithms for calculating the relevant directional derivatives can be adapted from those reported in Podinovski et al. (2016) and Roos, Terlaky, and Vial (2005).

Chambers and Färe (1998), we define T as *translation homothetic in the directions* (v, u) if

$$(x, y) \in T \Leftrightarrow (x + \alpha v, u + \alpha u) \in T, \alpha \in \mathbb{R}.$$

We have:

Proposition 7 – *$O_v^u(x, y) = I_x^v(x, y)$ for all (x, y) if and only if T is translation homothetic.*

Proposition 8 – *$O_v^u(x, y) = I_y^u(y, x)$ for all (x, y) if and only if T is translation homothetic.*

The usefulness of O_v^u as a measure of the translatability of T highlights an important difference between it and the directional input and output inefficiency measures. Those latter functions satisfy what we have called "Indication" that ensures that they characterize the set T. Signs differences aside, $O_v^u(x, y) \geq 0$ shares "one direction" of Indication because $(x, y) \in T \Rightarrow O_v^u(x, y) \geq 0$. But

$$O_v^u(x, y) \geq 0 \nRightarrow (x, y) \in T.$$

The reason is a natural consequence of the general properties of $O_v^u(x, y)$ established in:

Proposition 9 *a) $(x, y) \in T \Rightarrow O_x^u(x, y) \geq 0$; b) $O_v^u(x + \alpha v, y + \alpha u) = O_v^u(x, y)$ for $a \in \mathbb{R}$ (Translation Invariance); c) $O_v^u(x, y)$ is nondecreasing in the direction $(v, -u)$ (Monotonicity); d) $O_v^u(x, y)$ is concave in (x, y) (Concavity).*

Property b), Translation Invariance, implies that if one identifies an $(x, y) \in T$ for which $O_v^u(x, y) \geq 0$, then translating (x, y) in the direction (v, u) leaves measured inefficiency unchanged. If it is also true that

$$O_v^u(x, y) \geq 0 \Rightarrow (x, y) \in T,$$

that unchanged inefficiency then implies T is translation homothetic. But T exist that are not translation homothetic. Hence, $O_v^u(x, y) \geq 0 \nRightarrow (x, y) \in T$. One verifies this result visually using Figure 2. There $(\hat{x}, \hat{y}) \notin T$, but its associated overall inefficiency measure is positive.

Dual Characterizations of $O_v^u(x,y)$

We now study dual characterizations of $O_v^u(x,y)$. Define the *indicator function* for T, $\delta : \mathbb{R}^{N+M} \to \bar{R}$, as

$$\delta(x,y) = \begin{cases} 0 & \text{if } (x,y) \in T \\ \infty & \text{otherwise.} \end{cases} \tag{11}$$

For nonempty, convex T, $\delta(x,y)$ is proper, closed, and *sublinear* (positively homogeneous and convex). The cost, revenue, and profit functions are all conjugate functions of $\delta(x,y)$. Let (w,p) denote the input and output prices. The *cost function*, $c : \mathbb{R}^{N*} \times \mathbb{R}^{M} \to \bar{\mathbb{R}}$, is given by

$$c(w,y) = \inf_x \{w'x + \delta(x,y)\}; \tag{12}$$

the *revenue function* $R : \mathbb{R}^{M*} \times \mathbb{R}^{N} \to \bar{\mathbb{R}}$ by

$$R(p,x) = \sup_y \{py - \delta(x,y)\}; \tag{13}$$

and the *profit function*, $\pi : \mathbb{R}^{N*} \times \mathbb{R}^{M*} \to \bar{\mathbb{R}}$, by

$$\pi(w,p) = \sup_{x,y} \{p'y - w'x - \delta(x,y)\}. \tag{14}$$

The cost function is proper, closed, and superlinear in w, while the profit and revenue functions are proper, closed, and sublinear in prices. For convex T, the cost function is convex as a function of y, and the revenue function is concave as a function of x. Each dual function is conjugate dual to $\delta(x,y)$. (See, for example, Rockafellar (1970, Theorem 13.2).) To avoid repeating arguments, in what follows, we only consider the conjugate relation between $\pi(w,p)$ and $\delta(x,y)$ given by (14) and

$$\delta(x,y) = \sup_{(w,p)} \{p'y - w'x - \pi(w,p)\} \tag{15}$$

The next lemma summarizes known results on the relationships between the directional inefficiency measures and the dual characterizations of T (Luenberger 1992; Chambers, Chung, and Färe 1996):

Lemma 10

$$\begin{aligned} I_x^v(x,y) &= \sup_w \{c(w,y) - w'x : w'v = 1\} \\ &= \sup_{w,p} \{p'y - w'x - \pi(w,p) : w'v = 1\} \end{aligned}$$

$$\begin{aligned} I_y^u(y,x) &= \sup_p \{py - R(p,x) : p'u = 1\} \\ &= \sup_{w,p} \{py - wx - \pi(w,p) : p'u = 1\} \end{aligned}$$

Together Lemma 10 and Propositions 3 and 4 give:

Proposition 11

$$\begin{aligned} O_v^u(x,y) &= \sup_\phi \inf_w \{\phi + wx - c(w, y+\phi u) : w'v = 1\} \\ &= \sup_\beta \inf_p \{\beta + R(p, x-\beta v) - p'y : p'u = 1\} \\ &= \sup_\phi \inf_{w,p} \{\phi + \pi(w,p) - p(y+\phi u) + w'x : w'v = 1\} \\ &= \sup_\beta \inf_{w,p} \{\beta + \pi(w,p) - py + w(x-\beta v) : p'u = 1\} \qquad (16) \end{aligned}$$

Each expression in Proposition 11 manifests the same principle. $O_v^u(x,y)$ is the value of a saddle-point solution to a program that maximizes a concave function of directional movement in input or output space while minimizing a convex function (over (w,p)) measuring the difference between optimal performance (the cost function, the revenue function, or the profit function) and the decision maker's performance (cost, revenue, or profit) for hypothetical shadow prices. Ray (2007) and Aparicio et al. (2013) frame the difference between actual and optimal performance as "foregone profit". Proposition 11 shows that their results extend to measure "foregone" revenue and cost.

The saddle-point nature of $O_v^u(x,y)$ is an immediate consequence of the dual formulation of Ray's (2007) original problem for general T as:[6]

$$\begin{aligned} O_v^u(x,y) &\equiv \inf_{w,p} \{\pi(w,p) - p'y + w'x : p'u = 1, w'v = 1\} \\ &= \inf_{w,p} \sup_\lambda \{\gamma + \pi(w,p) - p'(y+\gamma u) + w'x : w'v = 1\} \\ &= \inf_{w,p} \sup_\lambda \{\lambda + \pi(w,p) - p'y + w'(x-\lambda v) : p'u = 1\}, \qquad (17) \end{aligned}$$

[6]Ray (2007) works in a DEA setting and requires $p'y = 1$ and $w'x = 1$. See his expressions (12) and (12a). Also see expression (28) in Aparicio et al. (2013).

where γ and λ are Lagrange multipliers. Convexity ensures the equivalence of (16) and (17).

In the absence of the price-normalizing constraints, Ray's (2007) problem reduces to (15), and its solution value is $-\delta(x, y)$. As the *supremum* of a superlinear function, $\delta(x, y)$'s "on-off" nature has the analytic advantage that it signals whether $(x, y) \in T$ but that same structure limits it usefulness for practical measurement. Figure 3 illustrates. By (14),

$$\delta(x, y) \geq p'y - w'x - \pi(w, p) \quad \forall \ (x, y, w, p).$$

Because $(x, y) \in T \Rightarrow \delta(x, y) = 0$, any feasible (x, y) must be associated with a (dual) hyperplane majorized by the graph of $\pi(w, p)$, which is partially depicted by the convex surface $0ABC$. Nonfeasible input-output combinations, on the other hand, are associated with hyperplanes majorizing $0ABC$ for which $p'y - w'x - \pi(w, p) > 0$. The homogeneity property of profit functions ensures that negative differences can be made arbitrarily small and positive differences arbitrarily large, hence expression (15).

Choosing a numeraire to normalize prices resolves the boundedness problem in practical settings. One can choose normalizations that result in an inefficiency measure that satisfies Indication. For example, if one only imposes $w'v = 1$ in (17), the input direction is the numeraire and the feasible region in Figure 3 is illustrated by the plane that passes through $\left(\frac{1}{v}, 0, 0, \right)$ parallel to the (p, π) plane (not drawn). And by Lemma 10, that choice generates the directional input inefficiency measure. Similarly, using u as the numeraire gives the directional output measure. Imposing both results in $O_v^u(x, y)$. That observation suggests that a natural interpretation of the three inefficiency measures is as *price-restricted indicator functions.*

Ray (2007, p. 233) does not work with indicator functions, so his motivation for normalization arises from different concerns. He measures foregone profit as the difference between maximal profit at shadow prices (w, p) that ensure

$$p'y - w'x = 0.$$

Nevertheless, the practical effect in Ray's (2007) formulation is the same because introducing a homogeneous constraint in a linear-programming setting permits unbounded solutions. His solution is to achieve the zero-profit condition by requiring $p'y = 1$ and $w'x = 1$, which in economics terms is equivalent to choosing two commodity bundles to define the numeraire.[7] This solves the computational issue associated with a homogenous constraint and permits different proportional movement in input and outputs, but it also requires *that equi-proportional changes in* (x, y) *in the direction of the numeraire bundles* v *and* u *must cancel one another out.* The result is a translation-invariant O_v^u.

A finite solution to (16) and (17) exists if and only if there exist (w, p, β, ϕ) satisfying

$$\begin{aligned} 1 - w'v &= 0 && (18) \\ \partial_w \pi(w, p) + x - \beta v &\ni 0 && (19) \\ \partial_p \pi(w, p) - y - \phi u &\ni 0 && (20) \\ 1 - p'u &= 0. && (21) \end{aligned}$$

Expressions (18) and (21) require that the shadow values of the "directions" u and v equal one. Expressions (19) and (20) reiterate *Hotelling's Lemma* and require that the translates of (x, y), $(x - \beta v, y + \phi u)$, are profit maximizing at (w, p, β, ϕ).

Multiplying (19) by optimal w and (20) by optimal p gives

$$\begin{aligned} \beta &= w'(x - x(w, p)) \\ \phi &= p'(y(w, p) - y) \text{ and} \\ O_v^u(x, y) &= \beta + \phi, \end{aligned}$$

where $x(w, p) \in -\partial_w \pi(w, p)$ and $y(w, p) \in \partial_p \pi(w, p)$ are profit-maximizing demand and supply vectors evaluated at the solution (w, p). The connection between $O_v^u(x, y)$ and the shadow values associated with its dual formulation reflects the superlinearity of that problem's objective function. Moreover, it reinforces the interpretation of O_v^u as a restricted indicator function.

Because $\frac{w'(x - x(w,p))}{O_v^u(x,y)}$ and $\frac{p'(y - y(w,p))}{O_v^u(x,y)}$ sum to one,[8] we can rewrite (19 and 20) as requiring

$$(x, y) + O_v^u(x, y)(-\alpha v, (1 - \alpha) u) \in (-\partial_w \pi(w, p), \partial_p \pi(w, p)) \quad \alpha \in \mathbb{R}, \tag{22}$$

[7] While multiple numeraire are uncommon in economics, they are commonplace in linear-programming formulations of the inefficiency measurement problem. Examples include the well-known Pareto-Koopmans measure suggested by Charnes, Cooper, Golany, Seiford, and Stutz (1985).

[8] They can differ in sign.

at an optimum. In Ray's (2007) terminology, the *endogenous projection(s)*[9] of (x, y) onto the profit-maximizing frontier, $(-\partial_w \pi(w,p), \partial_p \pi(w,p))$, occurs where the sum of (x, y) with a radial projection of an element of the line segment passing through $-v$ and u is profit maximizing. The length of that radial projection is determined by $O_v^u(x, y)$. Figure 4 illustrates. There the curve labelled $\bar{\pi}$ in the northeast quadrant is the isoprofit contour for the shadow-price solution (w, p). The normal to its tangent hyperplane at the solution gives minus the profit-maximizing demand vector and the profit maximizing outputs for those shadow prices and is labelled $(\partial_w \pi, \partial_p \pi)$. The relation between $(-x, y)$, the endogenous direction vector, and $(\partial_w \pi, \partial_p \pi)$ is portrayed in the northwest quadrant.[10] There the dotted line segment that passes through $-O_v^u v$ and $O_v^u u$ portrays the potentially optimal endogenous directions.

Concluding Remarks

We studied the overall inefficiency measure for convex technologies. We showed that the measure admits an alternative interpretation as measuring the maximal input (output) inefficiency consistent with projecting an observed input-output bundle by the same proportion in predetermined directions. We used that interpretation to demonstrate its relation to "translation elasticity" measures, to show that the measure reduces to a directional input (output) inefficiency measure for translation-homothetic technologies, to characterize O_v^u's behavior in (x, y), and to develop dual versions of the measure for general convex technologies. We showed that the overall inefficiency score corresponds to the saddlepoint of a problem that computes a " price-restricted indicator function".

[9] Recall $(-\partial_w \pi(w,p), \partial_p \pi(w,p))$ is a correspondence.

[10] In the two-dimensional case, the optimal shadow prices are determined by the normalizing constraints. We ignore this detail in depicting the solution in Figure 4.

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Appendix: Proofs

Proof. Lemma 1 and Lemma 2: We only prove Lemma 2, the proof for Lemma 1 is parallel. Properties b) and d) are standard (Luenberger 1992; Chambers, Chung, and Färe 1996, 1998). For a) $\Leftarrow$ follows by definition because $(x, y - 0u)$ is feasible. $\Rightarrow$ By definition, $\left(x, y - I_y^u(y,x)\,u\right) \in T$, and goodness in the direction $(v, -u)$ implies $(x, y) \in T$ for $I_y^u(y,x) \leq 0$. For c), by definition $\left(x, y - I_y^u(y,x)\,u\right) \in T$. Goodness in the direction $(v, -u)$ implies $\left(x + \beta v, y - \mu u - I_y^u(y,x)\,u\right) \in T$ for $\mu, \beta \geq 0$ and the result is a consequence of the definition of an *infimum*. ■

Proof. Propositions 2. We only prove Proposition 3, the proof for Proposition 4 is parallel. To show (5):

$$\begin{aligned}
O_v^u(x,y) &= \sup\{\beta + \phi : (x - \beta v, y + \phi u) \in T\} \\
&= \sup\{\phi + \sup\{\beta : (x - \beta v, y + \phi u) \in T\}\} \\
&= \sup\{\phi - inf\{\theta : (x + \theta v, y + \phi u) \in T\}\} \\
&= \sup\{\phi - I_x^v(x, y + \phi u)\}
\end{aligned}$$

The second equality follows by Bellman's Principle, the third by changing variables, and the fourth from the definition of I_x^v. Lemma 1.b implies

$$\phi - I_x^v(x, y + \phi u) = -I_x^v(x + \phi v, y + \phi u),$$

which establishes (6). ■

Proof. Propositions: 5 and 6. The proof is for Proposition 5. We first show (10) and then show that (10) implies (9). $\Rightarrow$ By Proposition 3,

$$O_v^u(x,y) = -I_x^v(x + \phi^* v, y + \phi^* u)$$

if and only if

$$I_x^v(x + \phi v, y + \phi u) - I_x^v(x + \phi^* v, y + \phi^* u) \geq 0 \quad \forall \phi.$$

For $\phi > \phi^*$, dividing both sides of this inequality by $\phi > \phi^*$ and taking a one-sided limit gives

$$lim_{\phi - \phi * \downarrow 0} \frac{I_x^v(x + \phi v, y + \phi u) - I_x^v(x + \phi^* v, y + \phi^* u)}{\phi - \phi^*} \geq 0,$$

or

$$D_{(x,y)}^{(v,u)} \circ I_x^v (x + \phi^* v, y + \phi^* u) \geq 0.$$

Reversing the sign $\phi - \phi^*$, similar arguments show that

$$-D_{(x,y)}^{-(v,u)} \circ I_x^v (x + \phi^* v, y + \phi^* u) \leq 0.$$

establishing necessity.

To show (9), use Lemma1.b (Translation) to rewrite

$$lim_{\phi - \phi * \downarrow 0} \frac{I_x^v (x + \phi v, y + \phi u) - I_x^v (x + \phi^* v, y + \phi^* u)}{\phi - \phi^*} \geq 0,$$

as

$$lim_{\phi - \phi * \downarrow 0} \frac{\phi^* - \phi + I_x^v (x, y + \phi u) - I_x^v (x, v, y + \phi^* u)}{\phi - \phi^*} \geq 0,$$

and the result follows immediately.

$\Leftarrow$ Convexity of I_x^v ensures that the minimum(a) at ϕ^* implied by the condition is global.

■

Proof. Propositions 7 and 8: We prove Proposition 7. By definition

$$\begin{aligned} O_v^u (x, y) &= \sup \{\beta + \phi : (x - \beta v, y + \phi u) \in T\} \\ &= \sup \{\beta + \phi : (x - (\beta + \phi) v, y) \in T\} \end{aligned} \tag{23}$$

Expression (23) follows by translation homotheticity and equals $-I_x^v (x, y)$. To go the other way, suppose that by hypothesis and Proposition 3,

$$-I_x^v (x, y) \geq \beta - I_x^v (x, y + \beta u) \quad \forall \beta \in \mathbb{R},$$

whence

$$I_x^v (x, y + \beta u) - I_x^v (x, y) \geq \beta \;\; \forall \beta \in \mathbb{R}.$$

Because β can be either positive or negative, the last inequality and Lemma 1.b imply

$$I_x^v (x + \beta v, y + \beta u) = I_x^v (x, y) \;\; \forall \beta \in \mathbb{R},$$

and Lemma 1.a gives the result. ■

Proof. Proposition 9 a) $(x, y) \in T \Rightarrow I_x^v(x, y) \leq 0$ by Lemma 1. Hence

$$\inf_{\phi} \{I_x^v(x + \phi v, y + \phi u)\} \leq I_x^v(x, y) \leq 0,$$

and the result follows by the definition of an *infimum.* b)

$$\begin{aligned} -O_v^u(x + \alpha v, y + \alpha u) &= \sup\{\beta + \phi : (x + \alpha v - \beta v, y + \alpha u + \phi u) \in T\} \\ &= \sup\{\beta + \phi : (x - (\beta - \alpha) v, y + (\alpha + \phi) u) \in T\} \\ &= \sup\{(\beta - \alpha) + (\alpha + \phi) : (x - (\beta - \alpha) v, y + (\alpha + \phi) u) \in T\} \end{aligned}$$

c) Let ϕ^* solve (6). Then

$$I_x^v(x + \alpha v + \phi^* v, y - \alpha u + \phi^* u) \leq I_x^v(x + \phi^* v, y + \phi^* u)$$

for $\alpha \geq 0$ by Lemma 1.c, and the conclusion follows immediately. d) follows by Lemma 1. d. ■

Appendix: The Translation Elasticity

Balk, Färe, and Karagiannis (2015) treat a technology with frontier $\{(x, y) : F(x, y) = 0\}$. By Lemma 1 both I_x^v and I_y^u are potential candidates for F. We treat the case where $F(x, y) \equiv I_x^v(x, y)$. Let $\alpha \in \mathbb{R}$ and define $\mu(\alpha, x, y)$ as the implicit solution to

$$I_x^v(x + \alpha v, y + \mu(\alpha, x, y) v) = 0$$

with $\mu(\alpha, x, y) \to 0$ as $\alpha \to 0$ if $I_x^v(x, y) = 0$. By convexity, I_x^v is almost everywhere differentiable. Balk et al. (2015) define the *translation elasticity*, $\epsilon_v^u(x, y)$, as

$$\epsilon_v^u(x, y) \equiv \frac{\partial \mu(\alpha, x, y)}{\partial \alpha}|_{\alpha \to 0}.$$

Performing the indicated differentiation gives

$$\epsilon_v^u(x, y) = -\frac{D_x^v \circ I_x^v(x, y)}{D_y^u \circ I_x^v(x, y)}.$$

By Translation (Lemma 1. b)

$$I_x^v(x + \lambda v, y) = I_x^v(x, y) + \lambda$$

whence

$$\begin{aligned} D_x^v \circ I_x^v (x + \lambda v, y) &= \lim_{\lambda \to 0} \frac{I_x^v (x + \lambda v, y) - I_x^v (x, y)}{\lambda} \\ &= \frac{I_x^v (x, y) - \lambda - I_x^v (x, y)}{\lambda} \\ &= -1 \end{aligned}$$

Therefore,

$$\epsilon_v^u (x, y) = \frac{1}{D_y^u \circ I_x^v (x, y)}.$$

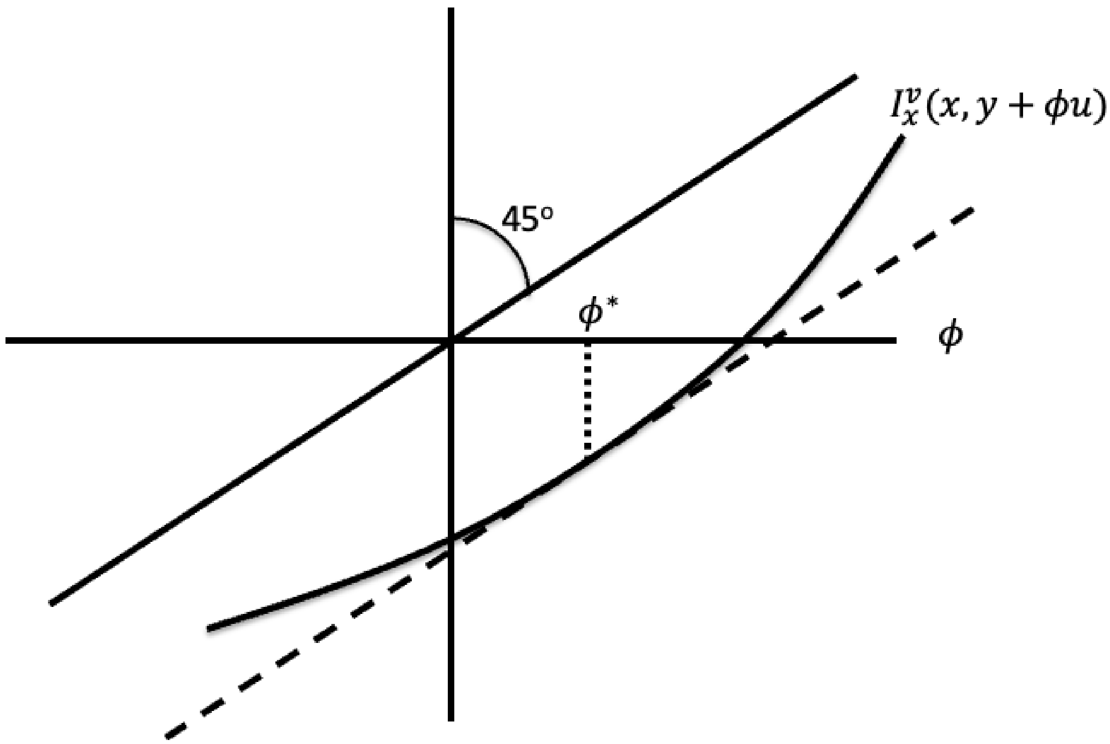

Figure 1: Overall Inefficiency Determined

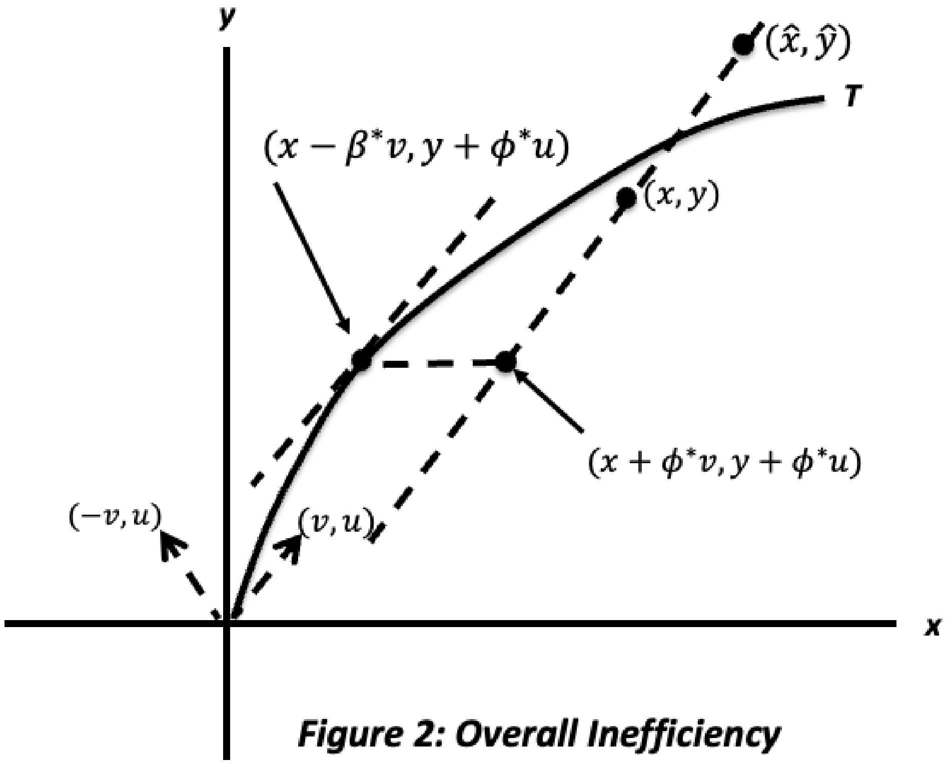

Figure 2: Overall Inefficiency

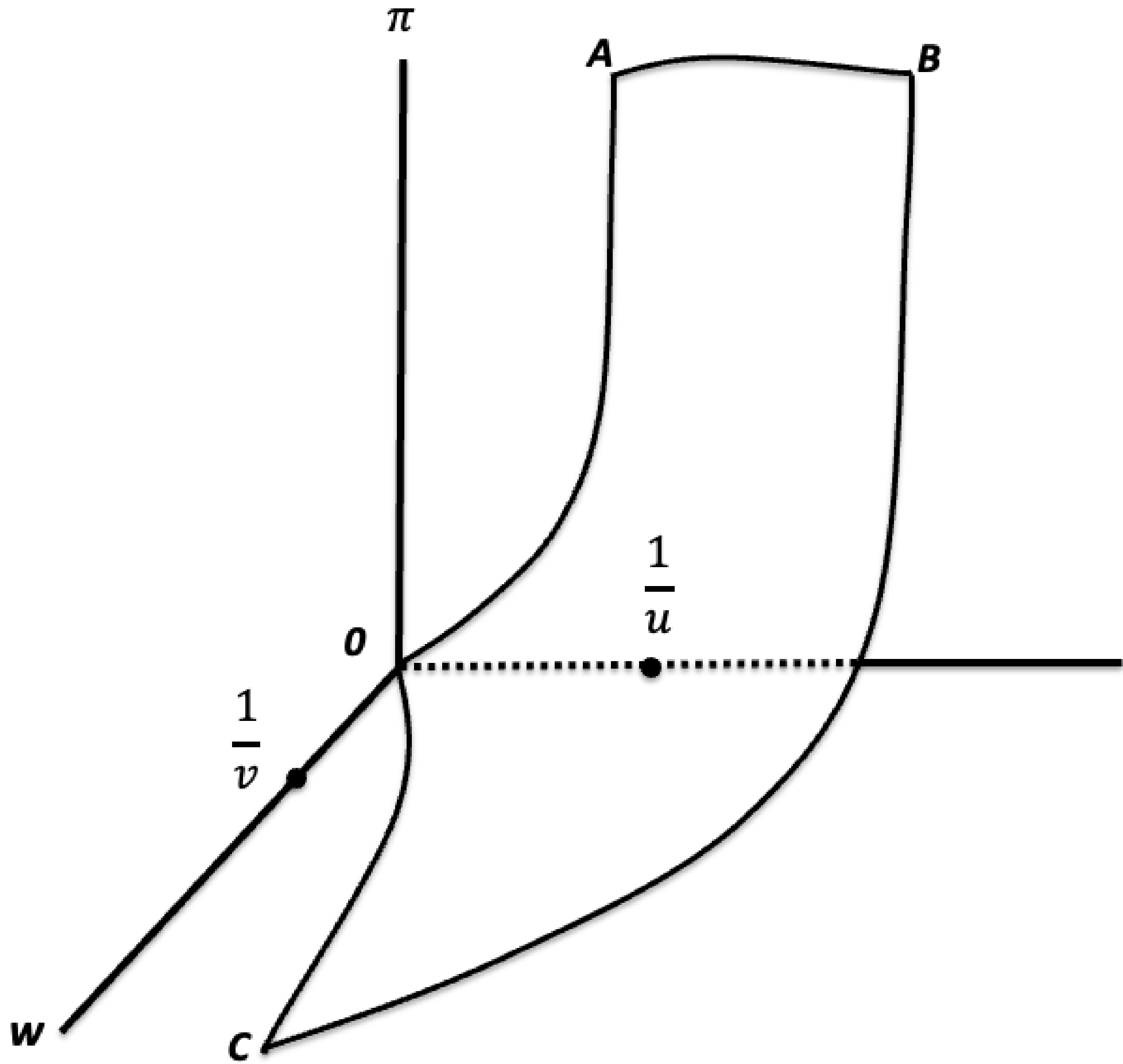

Figure 3: Restricted Indicator Function

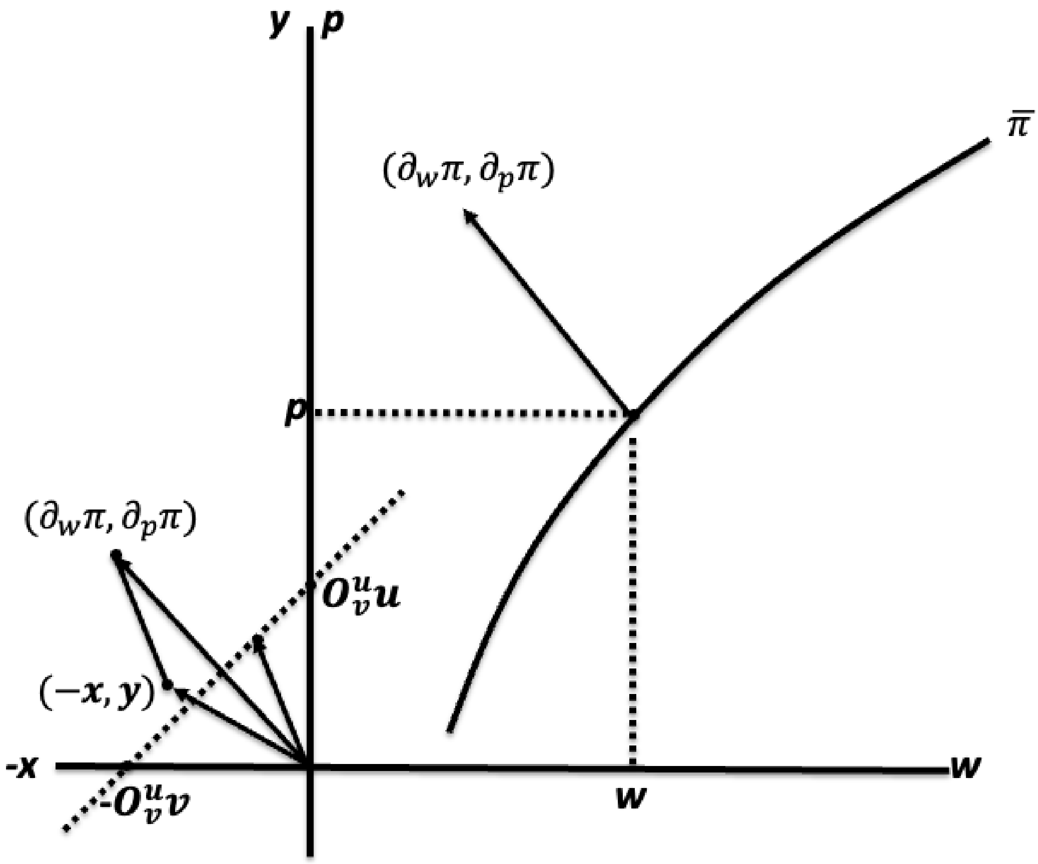

Figure 4: Endogenous Direction in the Dual Space