Numeraire Choice and Inefficiency Measurement

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1 Introduction

Evaluating the efficiency of a decision maker's performance requires an objective standard for comparison and a measurement tool. Farrell (1957) early developed methods for single-output, constant returns to scale technologies. Charnes, Cooper, and Rhodes (1978) extended Farrell's approach to mutiple-output technologies. Simultaneously, Färe and Lovell (1978) clarified the relationship between Farrell's efficiency measure and Shephard's (1953, 1970) distance functions. Banker, Charnes, and Cooper (1984) showed how to accommodate nonconstant returns in the Cooper et al. (1978) Data Envelopment Analysis (DEA) model, how to relate DEA to Shephard's (1970) axioms, and the relationship between DEA efficiency measures, Shephard's input (output) distance functions, and minimal cost (maximal revenue) functions.

These methods were ratio-based. Farrell (1957) measured inefficiency as actual cost divided by minimal cost, and Charnes et al. (1978) posed their problem as a fractional program. These measures proved attractive for comparing relative cost (revenue) performance and were widely applied. But similar approaches for relative profit performance remained elusive. A practical problem was the ratio-based criterion that struggled with negative or zero profits.

Following Nerlove (1965), Chambers, Chung, and Färe (1998) showed how to adapt Luenberger's (1992) shortage function to profit-based comparisons. The key ideas were to adopt a difference-based approach and to measure technical inefficiency in units of a predetermined input-output vector ("the direction"). The introduction of directional measures emphasized an inherent arbitrariness in measuring technical inefficiency. In that approach, the direction serves as the numeraire and determines the units of the inefficiency measure. But the directional choice resides with the analyst and remains arbitrary. That arbitrariness soon prompted a raft of competing inefficiency measures (see, for example, Briec and Lesourd 1999; Pastor, Ruiz, and Sirvent 1999; Portela and Thanassoulis 2007; Ray 2007; Cooper, Pastor, Aparicio, and Borras 2011; Zofio, Pastor, and Aparicio 2013).

We study the objective choice of inefficiency measures. Our analysis uses a generalization of a "price-based" approach pioneered by Charnes et al. (1978) and Ray (2007) to induce an objective technical inefficiency measure. The approach seeks "real shadow prices" that make a decision maker's performance as economically efficient as possible. Our focus is on how the numeraire choice affects the point on the efficient frontier to which the decision maker's performance is compared, the induced technical inefficiency measure, and whether the induced inefficiency measure offers a cardinal representation of the technology. The analysis is for general technologies, but we illustrate its applicability to the polyhedral setting that is a cornerstone of many applied studies.

We show that a natural complementarity exists between the chosen numeraire, the technical inefficiency measure, and the ability of that measure to characterize the technology. These results permit inferences on the objective identification of inefficiency measures and cardinal representations of technologies that accommodate known shortcomings of the canonical free disposal hull model. We use these results to introduce an inefficiency measure, the *polyhedral inefficiency measure*, that encompasses a broad array of existing measures as special cases.

Section 2 introduces notation and the model. Section 3 treats the problem of isolating the point(s) on the technology's frontier used to evaluate a decision maker's performance and to define a technical inefficiency measure. Section 4 details the properties of the derived inefficiency measure and develops conditions required for the inefficiency measure to be a cardinal representation of the underlying technology. Section 5 introduces the polyhedral inefficiency measure and examines some of its different manifestations. Section 6 concludes.

2 Notation and the Model

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For $f : \mathbb{R}^S \to \bar{\mathbb{R}}$ define its *subdifferential correspondence*, $\partial f : \mathbb{R}^S \rightrightarrows \mathbb{R}^{S*}$, by

$$\partial f(x) \equiv \left\{ q \in \mathbb{R}^{S*} : q'(z - x) \leq f(z) - f(z) \right\},$$

and its (one-sided) *directional derivative* in the direction v, $f^v : \mathbb{R}^S \to \mathbb{R}$, by

$$f^v(x) \equiv lim_{\mu \downarrow 0} \frac{f(x + \mu v) - f(x)}{\mu}.$$

Define the *indicator function*, $\delta : \mathbb{R}^S \to \{0, \infty\}$, for $X \subset \mathbb{R}^S$ by

$$\delta(x|X) = \begin{cases} 0 & \text{if } x \in X \\ \infty & \text{otherwise.} \end{cases} \tag{1}$$

Expression (1) shows that the indicator function characterizes X in the sense that knowledge of X and it are equivalent. In what follows, we refer to it as an *Indication Property.*[1] $\delta(x|X)$ is proper,[2] closed,[3] and sublinear (positively homogeneous and convex) as a function of x if X is closed, nonempty, and convex.

We model feasible production outcomes (the technology) by $Z \subset \mathbb{R}^S$ that is nonempty, closed, and convex. To preserve generality and notational simplicity, we treat production outcomes as "netputs". The traditional distinction between inputs and outputs is available by setting $z = (-x, y)$ where $x \in \mathbb{R}^N_+$ denotes inputs and $y \in \mathbb{R}^M_+$ denotes outputs. The *recession (asymptotic) cone for* Z is:

$$Z_\infty \equiv \left\{ d \in \mathbb{R}^S : Z + \beta d \subset Z, \beta \geq 0 \right\}.$$

Z admits different interpretations. For example, it could represent a technology set, an output set, or an input set. For concreteness sake, we adopt the technology-set metaphor. We can relax the convexity assumption on Z. But because we base our measure on a comparison with what is profit maximizing, little generality loss results from restricting attention to convex structures. Nonconvex technology sets are economically indistinguishable from their convex hulls. (Example 6 below illustrates).

[1] This use of the term, Indication, differs from that in the axiomatic inefficiency-measurement literature (for example, Russell and Schworm 2009). There, "Indication" means that if an inefficiency index takes on a critical value, typically one or zero, it signals that the observation is technically efficient.

[2] A convex function f is *proper* if its effective domain,

$$dom(f) \equiv \left\{ x \in \mathbb{R}^{\mathbb{S}} : f(x) < \infty \right\}$$

is nonempty and $f(x) > -\infty$ for all x.

[3] A function is closed if its closure is the function itself. For proper convex functions, closedness is equivalent to lower semi-continuity (Rockafellar 1970, p. 52).

The *support function* for Z, $\delta^* : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, is

$$\begin{aligned} \delta^*(q|Z) &\equiv \sup\{q'z : z \in Z\} \\ &= \sup_{z\in\mathbb{R}^S}\{q'z - \delta(z|Z)\} \end{aligned} \tag{2}$$

δ^* is proper, closed, and sublinear as a function of $q \in \mathbb{R}^{S*}$, and by (2) it is the (convex) conjugate function of δ. We call it the *profit function*, but it is open to other interpretations. By standard results (for example, Rockafellar 1970):

$$\delta(z|Z) = \sup_{q\in\mathbb{R}^{S*}}\{q'z - \delta^*(q|Z)\} \tag{3}$$

$$z \in \partial\delta^*(q|Z) \Leftrightarrow q \in \partial\delta(z|Z) \quad \textit{(Shephard-Hotelling Lemma)} \tag{4}$$

Expression (3) reflects the dual conjugacy of $\delta(z|Z)$ and $\delta^*(q|Z)$. Expression (4), the Shephard-Hotelling Lemma in economics, says that z solves (2) if and only if q solves (3).

$Q \subset \mathbb{R}^{S*}$ is nonempty, closed, convex, and satisfies $0 \in Q$. We interpret $\mathbb{R}^{S*}$, $\mathbb{R}^S$'s dual space, as price space.[4] To accommodate the potential for "good" and "bad" netputs, prices can be either positive or negative. The *gauge function* for Q, $\gamma : \mathbb{R}^{S*} \to \bar{\mathbb{R}}$, is defined

$$\gamma(q) \equiv \inf\{\mu > 0 : q \in \mu Q\}. \tag{5}$$

$\gamma(q)$ is proper, closed, and sublinear as a function of q,[5] and provides a cardinal representation of Q (Rockafellar 1970)

$$Q = \left\{q \in \mathbb{R}^{S*} : \gamma(q) \leq 1\right\} \tag{6}$$

3 The Inefficiency Measure

We follow Nerlove (1965) and evaluate the decision maker's performance, $z \in \mathbb{R}^S$, using the difference between $z's$ value at prices, $q \in \mathbb{R}^{S*}$, and maximal profit

$$q'z - \delta^*(q|Z). \tag{7}$$

[4] $\mathbb{R}^{S*}$, of course, equals $\mathbb{R}^S$. We maintain the notational distinction to ensure a similar distinction between quantities and prices.

[5] As a sublinear function, $\gamma(q)$ also represents the support function for the closed, convex set $\left\{z \in \mathbb{R}^S : q'z \leq \gamma(q) \quad \forall q\right\}$.

We interpret (7) as (minus) *profit foregone* at prices q by operating at z.[6] By (3)

$$\delta(z|Z) \geq q'z - \delta^*(q|Z) \quad \forall (z, q) \qquad \textit{(Fenchel's Inequality)}$$

so that $z \in Z$ implies

$$0 \geq q'z - \delta^*(q|Z) \quad \forall q.$$

Foregone profit is positively homogeneous in q, reflecting its nominal nature. That positive homogeneity is reflected in the "either-or" nature of the indicator function as the *supremum* of a sublinear function. When $z \in Z$, the supremum is achieved at zero, but when $z \notin Z$, the supremum is arbitrarily large. Thus, $\delta(z|Z)$, as its name implies, "indicates" when z is technically feasible and when it is not. But it contains no information on where z falls in Z. To achieve such a measure, we convert from nominal (money) terms to real terms by deflating prices by a common numeraire. Specifically, we interpret $\gamma(q)$ as a price index[7] and use it to deflate prices and profit to obtain the (real) *economic inefficiency* (EI) measure, $EI : \mathbb{R}^{S*} \times \mathbb{R}^S \to \mathbb{R}$,

$$EI(q, z) \equiv \frac{q'z - \delta^*(q|Z)}{\gamma(q)} = -\frac{\delta^*(q|Z - z)}{\gamma(q)}, \tag{8}$$

where $Z - z$ denotes Z translated one unit in the direction $-z$. The positive homogeneity of $\gamma(q)$ ensures that $EI(q, z)$ is homogeneous of degree zero in q and, thus, invariant to rescaling prices. We call $\gamma(q)$ the *numeraire*.

Farrell (1957) decomposed inefficiency into a *technical inefficiency* component and an *allocative inefficiency* component. To generate an analogous decomposition, we choose $\hat{z} \in \partial\delta^*(\hat{q}|Z)$ for some $\hat{q}$ and then subtract zero in the form $q'(\hat{z} - \hat{z})$ from (8) to get

$$EI(q, z) = \frac{q'(z - \hat{z}) + q'\hat{z} - \delta^*(q|Z)}{\gamma(q)}. \tag{9}$$

[6] Foregone profit can be redefined to be positive. Our definition emphasizes the connection of our technical inefficiency measure to the conjugate dual functions δ and δ^*.

[7] There are different options and interpretations. For example, the ideal producer price index for a given bundle of inputs is defined by its revenue function, which is sublinear. Similarly, an ideal consumer price index is defined as an expenditure function for a reference utility (Konüs 1939; Malmquist 1953; Blackorby, Primont, and Russell 1978). A simpler option, which is the basis of the Paasche and Laspeyres indices and directional and radial distance measures (see Section 5), is to choose a fixed netput bundle $g \in \mathbb{R}^S$ and set $\gamma(q) = q'g$.

We judge the technical inefficiency (TI) of z using the first component of (9),

$$\frac{q'(z-\hat{z})}{\gamma(q)},$$

and its allocative inefficiency (AI) by the residual

$$\frac{q'\hat{z}-\delta^*(q|Z)}{\gamma(q)}.$$

TI measures the real value of the move from z to the frontier of Z in the direction $(\hat{z}-z)$ and AI measures the real value of the move from $\hat{z}$ to the efficient $\partial\delta^*(q|Z)$. Other possible decompositions, of course, exist. Accommodating them requires adjusting arguments, but the same principles apply.

To provide an objective basis for choosing $\hat{z}$, we follow Charnes et al. (1978) and Ray (2007) and isolate it using variational methods that identify the real price vector(s) for which real foregone profit at z,

$$\frac{q'z-\delta^*(q|Z)}{\gamma(q)},$$

is maximized.[8] We convert this fractional problem to a constrained one using the same normalization strategy as Charnes et al. (1978). We solve[9]

$$\text{(10)} \qquad \delta^Q(z|Z) \equiv \sup_{q\in\mathbb{R}^{S*}} \{q'z-\delta^*(q|Z) : \gamma(q)=1\}$$

Problem (10) resembles the conjugate problem (3) that recaptures $\delta(z|Z)$ from $\delta^*(q|Z)$, but it restricts the price search to ensure that the numeraire's value is one. The notation δ^Q evokes its interpretation as a *restricted indicator function* that permits cardinal inefficiency comparisons. Denote the solution set for (10) as:

$$q(z) \in \arg\sup\{q'z-\delta^*(q|Z) : \gamma(q)=1\}.$$

For $\delta^Q(z|Z)$ is finite, the necessary and sufficient conditions require

$$\text{(11)} \qquad q'z-\delta^{*q}(q(z)|Z)-\lambda\gamma^q(q(z)) \le 0 \quad \forall q\in\mathbb{R}^{S*}$$

[8] Charnes et al. (1978) maximized the ratio of real shadow revenue to real shadow cost, and Ray (2007) used a profit-based criterion.

[9] Here we assume that γ used in the definition of EI is the same as γ used in (10). Other choices are possible, so long as they maintain sublinearity. Choosing a different deflator than $\gamma(q)$ in (9) requires adjusting our ultimate decomposition. To avoid the accompanying proliferation of notation, we treat $\gamma(q)$ in (9).

and

$$0 \in z - \partial\delta^* (q(z)|Z) - \lambda\partial\gamma(q(z)) \tag{12}$$

$$0 = 1 - \gamma(q(z)) \tag{13}$$

where λ is a Lagrange multiplier (for example, Rockafellar (1970) Theorems 28.1 and 29.1 and their corollaries). Expression (11) requires that the directional derivative of the Lagrangean bifunction, taken with respect to q, be nonincreasing in all directions at an optimum. Expression (12) and the Shephard-Hotelling Lemma (4) require that an element of the correspondence, $z - \lambda\partial\gamma(q(z))$, maximizes profit for $q(z)$. Expression (13) repeats the numeraire normalization. The elements of $\partial\delta^*(q(z)|Z)$ are our candidates for $\hat{z}$. We follow Ray (2007) and call them *endogenous projections.*

Evaluating (11) at $q(z)$ and $-q(z)$ gives[10]

$$q(z)' z - \delta^*(q(z)|Z) \leq \lambda \leq q(z)' z - \delta^*(q(z)|Z)$$

so that $\lambda = \delta^Q(z|Z)$ at a solution. By standard results, optimal λ is the shadow value (an element of the subdifferential of δ^Q) of a perturbation in the numeraire constraint. It equalling the optimal value of the program manifests the superlinear nature of the objective function. It also reinforces the crucial role that the numeraire choice plays in converting the potentially unbounded programming problem defining $\delta(z|Z)$ into a cardinal inefficiency measure.

The endogenous projections satisfy

$$z - \delta^Q(z|Z)\,\partial\gamma(q(z)) \cap \partial\delta^*(q(z)) \neq \emptyset. \tag{14}$$

Example 1 *Let $\gamma(q) = \|q\|$, the standard Euclidean norm. Then almost everywhere, $\partial\gamma(q) = \frac{q}{\|q\|}$ and $\partial\gamma(0) = \left\{z \in \mathbb{R}^S : z'q \leq \|q\| \quad \forall q \in \mathbb{R}^{S*}\right\}$. Thus, its endogenous projec-*

[10]The positive homogeneity of δ^* and γ in q ensure that

$$\begin{aligned} \delta^{*q(z)}(q(z)|Z) &= \delta(q(z)|Z) \\ \gamma^{q(z)}(q(z)) &= \gamma(q(z)). \end{aligned}$$

tions satisfy (almost everywhere)

$$z-\delta^{Q}\left(z|Z\right)\frac{q\left(z\right)}{\|q\left(z\right)\|}\in\partial\delta^{*}\left(q\left(z\right)\right).$$

Example 2 *Let* $\gamma\left(q\right)=\max\left\{q_{1},q_{2},\ldots,q_{S}\right\}$. *Then*

$$\partial\gamma\left(q\right)=co\left\{e^{k}:k\in M_{q}\right\},$$

where $co\left\{X\right\}$ *denotes the convex hull of* $X\subset\mathbb{R}^{S}$, $e^{s}\in\mathbb{R}^{S}$ *is the sth element of the standard orthonormal basis, and* $M_{q}=\left\{k:q_{k}=\max\left\{q_{1},q_{2},\ldots,q_{S}\right\}\right\}$ *is the active-index set. An endogenous projection satisfies*

$$z-\delta^{Q}\left(z|Z\right)co\left\{e^{k}:k\in M_{q}\right\}\cap\partial\delta^{*}\left(q\left(z\right)\right)\neq\emptyset.$$

Remark 3 *Formulate (10) in Lagrangean terms as*

$$\delta^{Q}\left(z|Z\right)=\inf_{\lambda}\sup_{q}\left\{\lambda+q'z-\delta*\left(q\right)-\lambda\gamma\left(q\right)\right\},$$

where λ *is a Lagrangean multiplier. Standard results (for example, Rockafellar 1970, Theorem 28.4 and its corollary) and the preceding arguments ensure that*

$$\begin{aligned}\delta^{Q}\left(z|Z\right) &= \inf_{\lambda}\left\{\lambda+\sup_{q}\left\{q'z-\delta^{*}\left(q\right)-\lambda\gamma\left(q\right)\right\}\right\}\\ &= \inf_{\lambda,q}\left\{\lambda:z-\partial\delta^{*}\left(q\right)-\lambda\partial\gamma\left(q\right)\right\}.\end{aligned}$$

Following Ray (2007), δ^{Q} *represents a directional inefficiency measure with an "endogenous direction" selected from* $\partial\gamma\left(q\right)$.

As Example 2 illustrates, a solution requires that $z-\delta^{Q}\left(z|Z\right)\partial\gamma\left(q\right)$ and $\partial\delta^{*}\left(q|Z\right)$ overlap. Because both are correspondences, not all elements of both sets necessarily fall in their intersection. Therefore, some elements of $\partial\gamma\left(q\right)$ may provide a solution while others do not. To distinguish the elements of $\partial\gamma\left(q\left(z\right)\right)$ that are solutions, we denote them by $g\left(z\right)\in\mathbb{R}^{S}$.

Using the endogenous projections of z, we have

$$EI\left(q,z\right)=\frac{\overbrace{\delta^{Q}\left(z|Z\right)q'g\left(z\right)}^{TI}+\overbrace{q'\left(z-\delta^{Q}g\left(z\right)\right)-\delta^{*}\left(q\right)}^{AI}}{\gamma\left(q\right)}. \tag{15}$$

TI, which is the product of two parts, δ^Q and $\frac{q'g(z)}{\gamma(q)}$, measures the real economic value of the move from z to its endogenous projection, $\delta(q(z))$. AI measures the real market value of the adjustment from the endogenous projection to the economically efficient, $\partial\delta^*(q|Z)$. Figure 1 illustrates. Counting in units of z_2, -EI is given by the distance AC on the vertical axis, -TI by the distance AB, and -AI by the distance BC.

4 Properties of $\delta^Q(z|Z)$

We have:

Proposition 4 *a)* $z \in Z \Rightarrow \delta^Q(z|Z) \leq 0$; *b)*

$$\begin{aligned}\delta^*(q|Z) + \delta(q|Q) &\leq \sup_z \left\{q'z - \delta^Q(z|Z)\right\} \\ &\equiv \delta^{Q*}(q|Z);\end{aligned}$$

and c) $\delta^Q(z|Z)$ *is convex as a function of* z.

Proof. a) By Fenchel's Inequality, $\delta(z|Z) \geq q'z - \delta^*(q)$ for all (z,q). By definition $z \in Z \Rightarrow \delta(z|Z) = 0$ whence $z \in Z \Rightarrow 0 \geq q'y - \delta^*(q|Z)$ for all q and the result. b) By construction,

$$\begin{aligned}\varphi(z) &\equiv \sup_q \{q'z - \delta^*(q|Z) - \delta(q|Q)\} \\ &= \sup_q \{q'z - \delta^*(q|Z) : q \in Q\} \\ &\geq \delta^Q(z|Z). \qquad (16)\end{aligned}$$

The inequality follows because $\{q : \gamma(q) = 1\} \subset Q$. $\varphi(z)$ is the conjugate function of the proper, closed, convex function $\delta^*(q|Z) + \delta(q|Q)$, whence

$$\delta^*(q|Z) + \delta(q|Q) = \sup_z \{q'z - \varphi(z)\}$$

(Rockafellar 1970, Theorem 12.2). The claim follows from (16).

c) Let $\bar{q}$ be an optimizer for $\mu z^0 + (1-\mu)z^0, \mu \in (0,1)$. Because $\bar{q}$ is feasible, $\delta^Q(z^0|Z) \geq \bar{q}z^0 - \delta^*(\bar{q}|Z)$ and $\delta^Q(z^1|Z) \geq \bar{q}z^1 - \delta^*(\bar{q}|Z)$. Multiplying the first by μ, the second by $(1-\mu)$, adding, and using the definition of $\bar{q}$ gives

$$\mu\delta^Q\left(z^0|Z\right) + (1-\mu)\,\delta^Q\left(z^1|Z\right) \geq \delta^Q\left(\mu z^0 + (1-\mu)\,z^0|Z\right),$$

establishing the desired convexity. ■

Convexity (Proposition 4.c) ensures that $\delta^{Q}(z|Z)$ is continuous everywhere on the relative interior of its effective domain, $\left\{z : \delta^{Q}(z|Z) < \infty\right\}$. Proposition 4.b establishes that its conjugate function, $\delta^{Q*}(q|Z)$, bounds for the sum of the profit function for Z and the indicator function for Q. $\delta^{Q*}(q|Z)$ is closed and convex as a function of q (Rockafellar 1970 Theorem 12.2). Thus, for any q feasible for (10), $\delta^{*}(q|Z) \leq \delta^{Q*}(q|Z)$. The inequality between these two conjugate function manifests the one-sided nature of Proposition 4.a, which ensures that $z \in Z \Rightarrow \delta^{Q}(z|Z) \leq 0$ but not that $\delta^{Q}(z|Z) \leq 0 \Rightarrow z \in Z$. Figure 2 illustrates a closed, convex, and nonempty Z for which $\delta^{Q}(z|Z) < 0$, but $z \notin Z$. The figure illustrates a situation where continuous movement from the endogenous projection in the directions $z - \partial\delta^{*}(q(z)|Z)$ passes through regions outside Z. Hence, $z \notin Z$ can exist for which $\delta^{Q}(z|Z) < 0$.

The technical difficulty that Figure 2 illustrates is a bounded Z for which $Z_{\infty} = \{0\}$. Our assumptions permit this. We can ensure δ^{Q} that satisfies an Indication property by endowing Z with an appropriate recession cone. In production-economics parlance, we can assume Z satisfies an appropriate *disposability* property. Or, in inefficiency-analysis parlance, we can impose an appropriate *Inefficiency Postulate*. Expression (14) the choice of an Inefficiency Postulate that ensures Indication. Requiring movements from $\partial\delta^{*}(q|Z)$, which lies on the boundary of Z, in the directions[11] $-\partial\gamma(q(z)|Z)$ to remain in Z works.

Remark 5 *By Proposition 4, if $-\partial\gamma(q(z)) \in Z_{\infty}$ for all z,*

$$\delta^{Q}(z|Z) \leq 0 \Leftrightarrow z \in Z \quad \textit{(Indication)}.$$

The argument is as follows. By expression (14)

$$z - \delta^{Q}(z|Z)\,\partial\gamma(q(z)) \in \partial\delta^{*}(q(z)|Z),$$

and $\delta^{Q}(z|Z) \leq 0$ gives the claim. The obvious shortcoming is that $\partial\gamma(q(z))$ is determined endogenously. A priori, such a restriction can be vacuous For example, if $\gamma(q) = \|q\|$ as in Example 1, so that (almost everywhere) $\partial\gamma(q) = \frac{q}{\|q\|}$.

[11] Recall $\partial\gamma(q(z)|Z)$ is correspondence.

Nevertheless, Remark 5 highlights the natural complementarity between the numeraire choice and the class of technologies that can be characterized by the inefficiency measure that the numeraire induces. Except in unusual circumstances, Z's true structure is unknown. To accommodate that problem, Banker et al.'s (1984) *Minimum Extrapolation Postulate* requires that an empirical approximation to Z be the smallest subset of $\mathbb{R}^S$ containing the data that is consistent with accepted properties (postulates) of Z. In its original formulation, the Banker et al. (1984) *Inefficiency Postulate* posits that $Z_\infty = \mathbb{R}^S_-$, free disposability of inputs and outputs.[12]

It is well understood that free disposability of inputs and outputs can clash with the first law of thermodynamics and can be inconsistent with the presence of undesireable by-products, input-output congestion (see, for example, Ray 2004, pp. 176-185), and other characteristics of physical production processes. A vibrant literature treats the design of models to accommodate such problems. A principal focus is the appropriate form of an Inefficiency Postulate (see, for example, Podinovski and Kuosmanen (2011) and Murty and Russell (2022).) The import of Remark 5 and the surrounding discussion is that resolution of that issue can have important implications for the choice of a numeraire in designing an inefficiency measure.

Example 6 *Let there be T observations, $u^t \in \mathbb{R}^S$, on decision-maker performance. Denote the data cloud by*

$$U \equiv \left\{u^1, u^2, \ldots, u^T\right\}.$$

The profit function for U is

$$\begin{aligned} \delta^*(q|U) &= \sup_z \{q'z : z \in U\} \\ &= \sup \{q'z : z \in co\{U\}\} \\ &= \delta^*(q|co\{U\}) \end{aligned}$$

with

$$\partial\delta^*(q|Z) = co\left\{u^k : k \in M_U\right\}$$

[12]This Inefficiency Postulate is often maintained in axiomatic treatments of inefficiency measures (see, for example, Russell and Schworm 2009, 2011).

where

$$co\{U\} = \left\{ z \in \mathbb{R}^S : z = \sum_{t=1}^{T} \lambda_t u^t, \lambda_t \geq 0 \ \ \forall t, \sum_{t=1}^{T} \lambda_t = 1 \right\},$$

and $M_U = \left\{k : q'u^k = \delta^*(q|Z)\right\}$ *is an active-index set and* $co\left\{u^k : k \in M_U\right\} \subset co\{U\}$*. The minimum-extrapolation technology consistent with* U *and a profit-maximization postulate is* $co\{U\}$*.*[13] *Figure 3 illustrates* $co\{U\}$ *as the triangle with vertices at the data points* u^1, u^2*, and* u^3*. Because* $co\{U\}$ *is bounded, its recession cone is* 0 *and does not permit free disposability of inputs or outputs. Its boundedness implies that* $\delta^*(q|co\{U\})$ *is finite for all* $q \in \mathbb{R}^{S*}$ *(for example, Rockafellar 1970, Corollary 13.2.2). The canonical DEA technology is the free disposal hull of* $co\{U\}$*,*

$$F\{co\{U\}\} \equiv \left\{ z \in \mathbb{R}^S : z \leq \sum_{t=1}^{T} \lambda_t u^t, \lambda_t \geq 0 \ \ \forall t, \sum_{t=1}^{T} \lambda_t = 1 \right\}.$$

with recession cone $\mathbb{R}^S_-$*.* $co\{U\} \subset F\{co\{U\}\}$ *and, thus, is the "more conservative approximation" to* Z*. Because* $F\{co\{U\}\}$ *has a nontrivial recession cone,* $\delta^*(q|F\{co\{U\}\})$ *is not everywhere finite. In particular,* $\delta^*(q|F\{co\{U\}\}) = \infty$ *for* $q \in \mathbb{R}^S_-$ *and, generally,* $\delta^*(q|F\{co\{U\}\}) \geq \delta^*(q|co\{U\})$*. Points such as* u^3 *in Figure 3, associated with a negative shadow price for* z_1*, can belong to* $\partial\delta^*(q|co\{U\})$ *but they cannot belong to* $\partial\delta^*(q|F\{co\{U\}\})$*.*

Many studies do not treat Indication as a critical criterion for an inefficiency measure. The reasoning is that inefficiency measures are meant to study situations where the assumption that a common Z generated the data cloud is legitimate. Therefore, the analyst's only interest is the measure's ability to characterize or measure distance from a frontier, part a) of Proposition 4. For example, axiomatic derivations of inefficiency measures often restrict the domains for their derived measures to subsets of $\mathbb{R}^S$ that are feasible for technologies satisfying a prescribed set of regularity conditions (for example, Russell and Schworm 2009, 2011). Remark 5's takeaway message for such instances, and more generally for applied inefficiency measurement, is that the nexus between the numeraire choice and the requirements for Indication should guide formulation of the prescribed regularity conditions. At a minimum, it seems desireable that 'regularity' conditions should be consistent with our understanding of

[13]Hence, the Banker et al. (1984) Convexity Postulate and the profit maximization hypothesis identify the same minimum extrapolation of U.

the way the physical universe operates. Inefficiency measures requiring regularity properties inconsonant with the physical problem must remain problematic.

5 A polyhedral inefficiency measure

Figure 3 and Example 6 illustrate the practical problem of designing an inefficiency measure that accommodates a technology whose upper boundary (the production function) has regions where it is negatively sloped. Even the most basic economic formulation of a technology, the lazy-S shaped production function familiar to introductory-economics students, exhibits regions of negative marginal returns. The economic relevance of such regions is ruled out by demonstrating that an economically rational producer would never locate at such points. But the same argument shows that economically rational producers never produce inefficiently. We now study a class of gauge functions that provide some flexibility in accomodating such practical circumstances.

5.1 The Inefficiency Measure

Let $G = \left\{g^1, g^2, \ldots, g^K\right\}$ with each $g^k \in \mathbb{R}^S$ and

$$D \equiv \cap_{k=1}^{K} \left\{q \in \mathbb{R}^{S*} : q'g^k \leq 1\right\}.$$

As the intersection of a finite number of closed, convex half spaces, $D \in \mathbb{R}^{S*}$ is closed, convex, and contains 0. Its gauge is the polyhedral convex function

$$\gamma^D(q) = max\left\{q'g^1, q'g^2, \ldots, q'g^K\right\}$$

with

$$\partial\gamma^D(q) = co\left\{g_k : k \in M_G\right\},$$

where $M_G = \left\{k : q'g_k = \gamma^D(q)\right\}$ is an active-index set and $co\left\{g_k : k \in M_G\right\} \subset co\left\{G\right\}$.

We define a *polyhedral inefficiency measure* (for D) as

$$\delta^D(z|Z) \equiv \sup_q \left\{q'z - \delta^*(q|Z) : \gamma^D(q) = 1\right\}$$

Because $co\{g_k : k \in M_G\} \subset co\{G\}$, its endogenous projections must satisfy

$$z - \delta^D(z|Z)\left\{g \in \mathbb{R}^S : g = \sum_{k=1}^{K} \mu_k g^k,\ \forall \mu_k \geq 0, \sum_{k=1}^{K} \mu_k = 1\right\} \cap \partial\delta^*(q|Z) \neq \emptyset.$$

Thus, the polyhedral inefficiency measure is interpretable as a directional inefficiency measure that permits different proportional movements in K different directions. The different proportions are determined by the product of $\delta^D(z|Z)$ and the optimal selection variables (the $\mu_k's$) for the convex hull of M_G. Thus, δ^D generalizes the Ray (2007) and Aparicio et al. (2013) overall inefficiency measure to an arbitrary number of directions which have the ability to accommodate a variety of recession cones for Z.

Example 7

$$\begin{aligned}\delta^D(z|co\{U\}) &= \sup_q \left\{q'z - \delta^*(q|co\{U\}) : \gamma^D(q) = 1\right\} \\ &= \inf_\lambda \{\lambda : z - \lambda co\{G\} \in \partial\delta^*(q|co\{U\})\}\end{aligned}$$

Figure 4 illustrates for $co\{U\}$ *and* $G = \{g^1, g^2\}$. *The endogenous projection for* z *is* u^1. *Here* $\delta^D(z) < 0$.

We have:

Corollary 8 *: a)* $z \in Z \Rightarrow \delta^D(z|Z) \leq 0$*; b)* $\delta^D(z|Z)$ *is convex in* z*; c)* $\delta^D\left(z + \alpha \sum_k g^k|Z\right) = \delta^D(z|Z) + \alpha, \alpha \in \mathbb{R}$ *(Enumeration); and d) if* $-co\{G\} \subset Z_\infty$,

$$\delta^D(z) \leq 0 \Rightarrow z \in Z \quad \textit{(Indication).}$$

Proof. a) and b) are established in Proposition 4. d) is an immediate consequence of Remark 5 and the discussion in the text. To show b), note that by the Lagrangean formulation of (10)

$$\delta^D\left(z + \sum_k g^k|Z\right) = inf_{\lambda,q}\left\{\lambda : z + \alpha\sum_k g^k - \lambda co\{G\} \cap \partial\delta^*(q|Z) \neq \emptyset\right\}$$

But

$$z + \alpha\sum_k g^k - \lambda co\{G\}$$

equals

$$-\left\{\sum_{k=1}^{K}\mu_k\left((\lambda-\alpha)\,g^k-z\right),\ \forall\mu_k\geq 0,\sum_{k=1}^{K}\mu_k=1\right\},$$

whence

$$\delta^D\left(z+\sum_k g^k|Z\right) \quad = \quad inf_{\beta,q}\left\{\beta:-\left\{\sum_{k=1}^{K}\mu_k\left(\beta g^k-z\right),\ \forall\mu_k\geq 0,\sum_{k=1}^{K}\mu_k=1\right\}\cap\partial\delta^*\left(q|Z\right)\neq\emptyset\right\}+\alpha,$$

where $\beta\equiv\lambda-\alpha$. ■

Enumeration (Corollary 8.c) is the natural extension of the translation property of directional inefficiency measures to arbitrary dimensions.

5.2 Special Cases

We now relate $\delta^D(z|Z)$ to existing inefficiency measures that encompass both "path-based" and "slacks-based" inefficiency measures (Russell and Schworm 2017). The results are for arbitrary convex technologies even though some measures have only been specified for DEA models.

5.2.1 Path-based measures

Luenberger Shortage or Directional Distance : Letting K=1 and $g^1\in\mathbb{R}^S$ gives Luenberger's (1992) shortage function or its directional distance function analogue (Chambers, Chung, and Färe 1996, 1998) as δ^D. If $-g^1\in Z_\infty$, $\delta^D(z|Z)$ satisfies Indication. An historically important special case is *Allais's (1943) Disposable Surplus* for netput k obtained by setting K=1 and $g^1=e^k$. The shortage function provides one means to accommodate failures of free disposability. For example, if netputs 1, 2,...,J < S are freely disposable while the remaining netputs are not, one can set K=1 and $g^1=(1,1,\ldots,1,0,\ldots,0)$. Then $-g^1\in Z_\infty$ permits Indication without imposing overall free disposability.

Gauges, Radial Distance Function, Sub-vector gauges : Let K=1 and $g^1=z$ to induce, depending upon the interpretation of Z, δ^D as a Minkowski (1911) functional (gauge) or a radial distance function (Debreu 1951, Shephard 1953, Malmquist 1953). Requiring $\lambda Z\subset Z$ (for $\lambda\leq 1$ weak disposability of all z) ensures Indication. Depending upon the interpretation, special cases include Farrell (1957), McFadden (1978), Charnes et al. (1978), Briec (1997), Färe, He, Li, and Zelenyuk (2019), and others. Define the partition $z = (z^0, z^1)$ and take $g^1 = (z^1, 0)$ to obtain a sub-vector radial measure of inefficiency (Färe, Grosskopf, and Lovell 1988). Input and output distance functions are special cases.

Overall Inefficiency Measure : Let K=2 and $g^1 = (v, 0)$ where $v \in \mathbb{R}^N$ and $g^2 = (0, u)$ where $u \in \mathbb{R}^{S-N}$ (Ray 2007; Aparicio et al. 2013). The corresponding gauge is $\gamma^D(q) = max\{q^{1\prime}v, q^{2\prime}u\}$, where q^1, q^2 are conformable subvectors of q. Indication is satisfied if $-co\{u, v\} \subset Z_\infty$.

5.2.2 Slacks-based measures

Russell-type Measures : Set $K = S$ and $g^k = z_k e^k$ to induce an inefficiency measure with structure similar to the Russell measure introduced by Färe and Lovell (1978).[14] Indication is satisfied if $Z_\infty = \mathbb{R}^S_-$.

Pareto-Koopmans Measure : Let $K = S$ and $g^k = \frac{e^k}{S}$ for all k. The induced δ^D corresponds to the Pareto-Koopmans technical inefficiency measure introduced by Charnes, Cooper, Golany, Seiford, and Stutz (1985). The resulting measure is $-\frac{1}{S}\delta^*(1|Z - z)$. The Pareto-Koopmans measure is a profit function for Z translated one unit in the direction $-z$. Its commensurable form is obtained by setting $g^k = \frac{e^k}{z_k S}$ when all $z_k \neq 0$. Indication is satisfied if $Z_\infty = \mathbb{R}^S_-$.

Weighted Average Inefficiency Measure : Let $K = S$ and $g^k = \frac{e^k}{w_k}$, for $w_k > 0$, $k = 1, \ldots, S$. Special cases include weighted average score introduced by Lovell and Pastor (1995) and studied by Cooper, Pastor, Aparicio, and Borras (2011). The Fukuyama and Weber (2009) "directional slacks-based measure" is the special case that corresponds to $g^k = \frac{e^k}{d_k}$, where d^k are interpreted as directions. The similarity to the Russell-type and Pareto-Koopmans measures is apparent.

[14]Many studies refer to such measures as Russell measures. But Russell and Schworm (2009) refer to them as Färe-Lovell measures. Hence, our "Russell-type" terminology.

Partial Slacks : Another way to accommodate the lack of free disposability in certain directions is to specify a Luenberger (1992) measure with a direction containing 0 elements. A slacks-based approach that permits different proportional adjustments in properly disposable regions is to take $K < S$ and specify (after appropriate choice of indices) $g^k = e^k, k = 1, \ldots, K$.

6 Concluding Remarks

For general technologies, we studied a “price-based” approach to generating inefficiency measures. We showed how the numeraire choice determines the endogenous projection, the technical inefficiency measure, and the restrictions on the technology’s recession cone required to ensure Indication. We showed that the induced technical inefficiency measure’s conjugate function bounds the sum of support function for Z and the indicator function for Q. We used these results to derive an inefficiency measure, the *polyhedral inefficiency measure*, that unites a broad array of existing measures under a single rubric that accommodates technologies with varying recession cones.

The demonstration that a broad range of practical inefficiency measures reduce to special cases of the polyhedral inefficiency measure implies that comparisons of their relative “reasonableness” reduce to to judgments of the “reasonableness” of their numeraires. Economists often treat normalization as a simple matter of setting one price equal to one. Such choices, however, can result in a inefficiency measures (the analogue of Allais’s disposable surplus) with properties that are unattractive to the inefficiency analyst. The general principle is that seemingly harmless assumptions to the economist can assume central importance to the inefficiency analyst. Unfortunately, the converse is also true, numeraire choices by the inefficiency analyst can render measurement and decomposition of EI irrelevant.

Our analysis contrasts with the axiomatic or “test” approach initiated by Färe and Lovell (1978).[15] That approach postulates axioms (properties) that an inefficiency index “should” possess for a technology that itself satisfies certain axioms (typically, free disposability, no land of Cockaigne, and bounded output sets). Hence, in spirit, it parallels decision theory.

[15] See Russell and Schworm (2009, 2011, 2017) for state-of-the-art treatments.

But where decision theory posits behavioral axioms, inefficiency axioms are less prosaic and more purely mathematical. For example, Färe and Lovell's (1978) original axioms were *Indication of Efficiency,*[16] *Monotonicity, and Homogeneity.* Moreover, where decision theory uses its axioms to deduce specific preference representations, axiomatic inefficiency analyses often treats its axioms as tests that pre-specified measures should pass.[17]

[16]See our earlier footnote on the Indication terminology.

[17]An exception is Chambers and Miller (2014) that uses an axiomatic approach to deduce classes of inefficiency measures.

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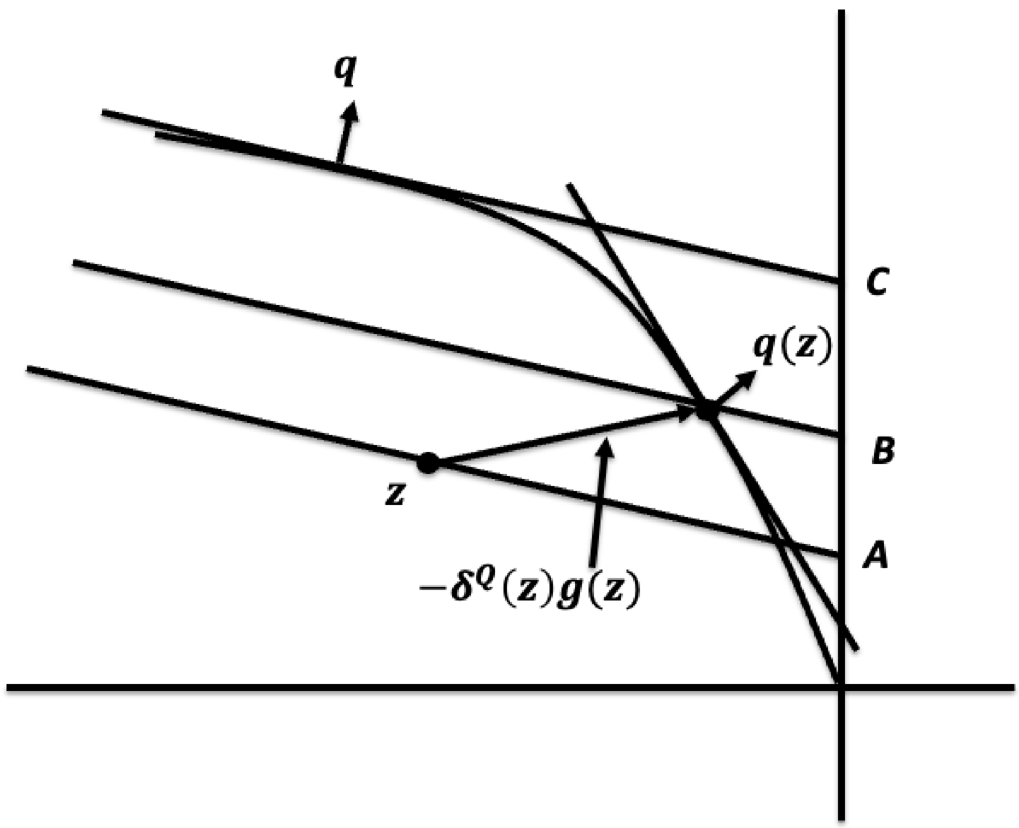

Figure 3: An Inefficiency Decomposition

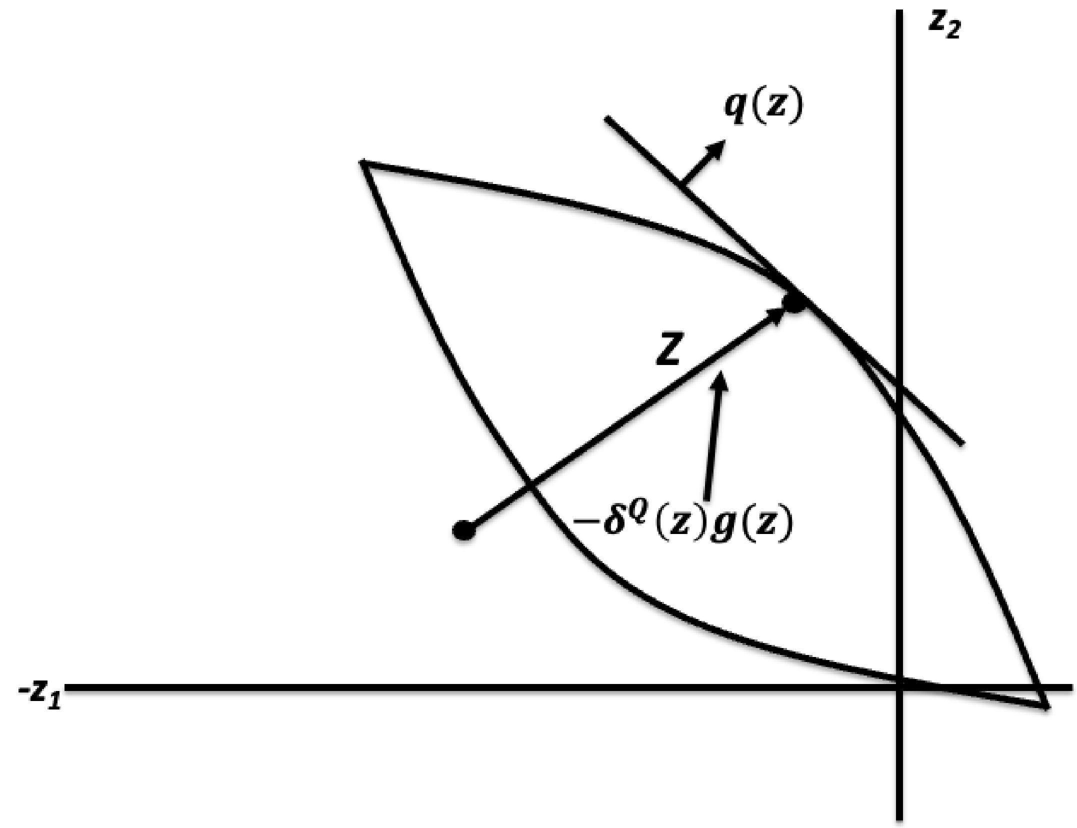

Figure 4: δ^Q is not a cardinal representation of Z

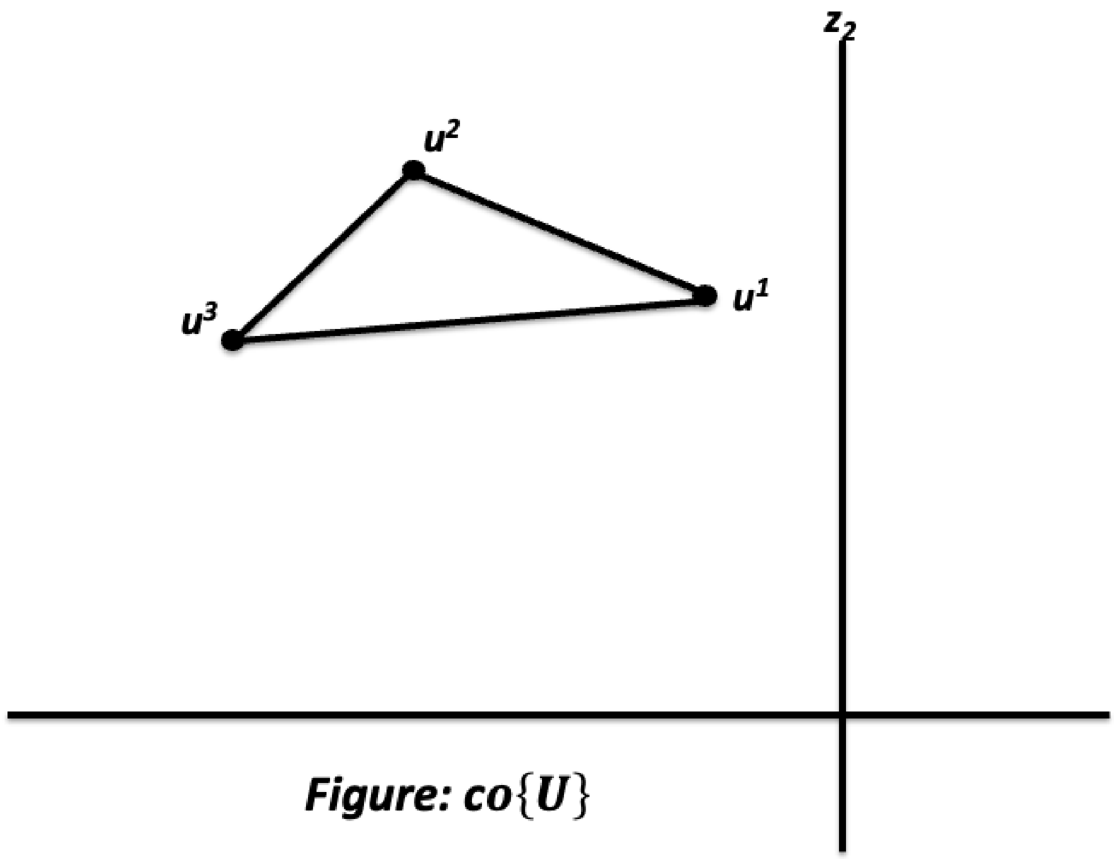

Figure: $co\{U\}$

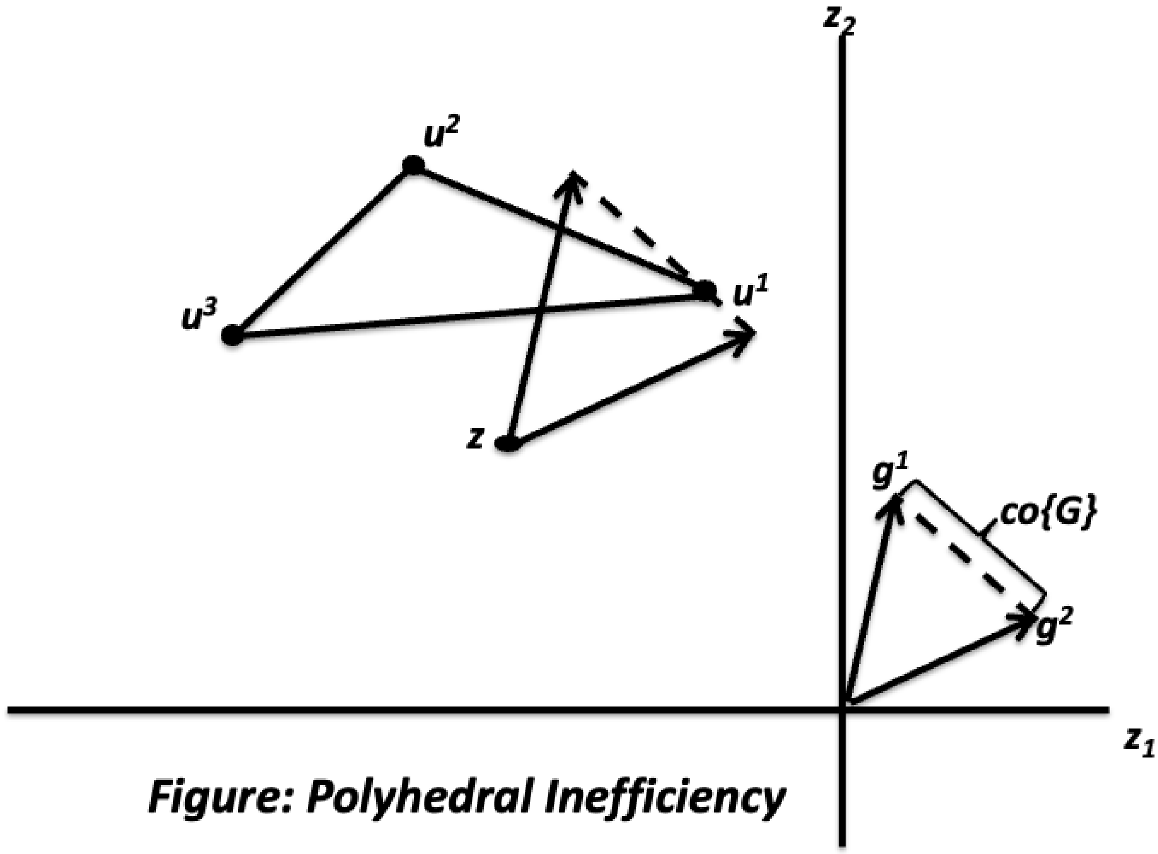

Figure: Polyhedral Inefficiency